Mathematical Excalibur

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Olympiad Corner

Below are the problems of the *2016 IMO Team Selection Contest I* for Estonia.

Problem 1. There are k heaps on the table, each containing a different positive number of stones. Jüri and Mari make moves alternatively; Jüri starts. On each move, the player making the move has to pick a heap and remove one or more stones in it from the table; in addition, the player is allowed to distribute any number of the remaining stones from that heap in any way between other non-empty heaps. The player to remove the last stone from the table wins. For which positive integers k does Jüri have a winning strategy for any initial state that satisfies the conditions?

Problem 2. Let p be a prime number. Find all triples (a,b,c) of integers (not necessarily positive) such that

$$a^b b^c c^a = p.$$

Problem 3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the equality $f(2^x+2y) = 2^y f(\,f(x)\,)\,f(y)$ for every $x,y\in\mathbb{R}$.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 21, 2016***.

For individual subscription for the next five issues for the 16-17 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2016

Kin Y. Li

This year Hong Kong served as the host of the International Mathematical Olympiad (IMO), which was held from July 6 to 16. Numerous records were set. Leaders, deputy leaders and contestants from 109 countries or regions participated in this annual event. A total of 602 contestants took part in this world class competition. Among the contestants, 71 were female and 531 were male.

After the two days of competition on July 11 and 12, near 700 contestants and guides from more than 100 countries or regions went to visit Mickey Mouse at the Hong Kong Disneyland for an excursion. That was perhaps the happiest moment in the IMO.

For Hong Kong, due to the hard work of the 6 team members and the strong coaching by Dr. Leung Tat Wing, Dr. Law Ka Ho and our deputy leader Cesar Jose Alaban along with the support of the many trainers and former team members, the team received 3 gold, 2 silver and 1 bronze medals, which was the best performance ever. Also, for the first time since Hong Kong participated in the IMO, we received a top 10 team ranking.

The Hong Kong IMO team members (in alphabetical order) are as follows:

(HKG1) Cheung Wai Lam, Queen Elizabeth School, Silver Medalist,

(HKG2) Kwok Man Yi, Baptist Lui Ming Choi Secondary School, Bronze Medalist,

(HKG3) Lee Shun Ming Samuel, CNEC Christian College, Gold Medalist,

(HKG4) Leung Yui Hin Arvin, Diocesan Boys' School, Silver Medalist,

(HKG5) Wu John Michael, Hong Kong International School, Gold Medalist and

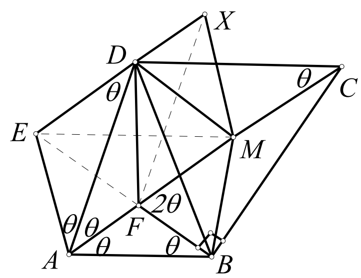
(HKG 6) Yu Hoi Wai, La Salle College, Gold Medalist.

The top 10 teams in IMO 2016 are (1) USA, (2) South Korea, (3) China, (4) Singapore, (5) Taiwan, (6) North Korea, (7) Russia and UK, (9) Hong Kong and (10) Japan.

The cutoffs for gold, silver and bronze medals were 29, 22 and 16 marks respectively. There were 44 gold, 101 silver, 135 bronze and 162 honourable mentions awardees.

Next, we will look at the problems in IMO 2016.

Problem 1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA=FB$ and F lies between A and C. Point D is chosen such that $DA=DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA=ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM||EX$ and $AE||MX$). Prove that lines BD, FX, and ME are concurrent.



From the statement of the problem, we get a whole bunch of equal angles as labeled in the figure. We have $\triangle ABF\sim\triangle ACD$. Then $AB/AC=AF/AD$. With, $\angle BAC=\theta=\angle FAD$, we get $\triangle ABC\sim\triangle AFD$.

(continued on page 2)

Then $\angle AFD=\angle ABC=90^\circ+\theta=180^\circ-\frac{1}{2}\angle AED$. Hence, F is on the circle with center E and radius EA. Then $EF=EA=ED$ and $\angle EFA=\angle EAF=2\theta=\angle BFC$. So B, F, E are collinear. Also, $\angle EDA=\angle MAD$ implies $ED||AM$. Hence E,D,X are collinear. From M is midpoint of CF and $\angle CBF=90^\circ$, we get $MF=MB$. Next the isosceles triangles EFA and MFB are congruent due to $\angle EFA=\angle MFB$ and $AF=BF$. Then $BM=AE=XM$ and $BE=BF+FE=AF+FM=AM=EX$. So $\triangle EMB\cong\triangle EMX$. As F and D lie on EB and EX respectively and $EF=ED$, we see lines BD and XF are symmetric respect to EM. Therefore, BD, XF, EM are concurrent.

Problem 2. Find all positive integer n for which each cell of an $n\times n$ table can be filled with one of the letters *I, M* and *O* in such a way that:

- in each row and each column, one third of the entries are *I*, one third are *M* and one third are *O*; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are *I*, one third are *M* and one third are *O*.

Note: The rows and columns of an $n\times n$ table are each labeled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integers (i, j) with $1\le i, j\le n$. For $n>1$, the table has $4n-2$ *diagonals* of two types. A diagonal of the first type consists of all cells (i, j) for which $i+j$ is a constant, and a diagonal of the second type consists of all cells (i,j) for which $i-j$ is a constant.

For $n=9$, it is not difficult to get an example such as

$$\begin{pmatrix} I & I & I & M & M & M & O & O & O\\ M & M & M & O & O & O & I & I & I\\ O & O & O & I & I & I & M & M & M\\ I & I & I & M & M & M & O & O & O\\ M & M & M & O & O & O & I & I & I\\ O & O & O & I & I & I & M & M & M\\ I & I & I & M & M & M & O & O & O\\ M & M & M & O & O & O & I & I & I\\ O & O & O & I & I & I & M & M & M \end{pmatrix}$$

For $n=9m$, we can divide the $n\times n$ table into $m\times m$ blocks, where in each block we use the 9×9 table above.

Next suppose a $n\times n$ table satisfies the conditions. Then n is a multiple of 3, say $n=3k$. Divide the $n\times n$ into $k\times k$ blocks of 3×3 tables. Call the center entry of the 3×3 tables a *vital entry* and call any row, column or diagonal passing through a vital entry a *vital line.* The trick here is to do double counting on the number N of all ordered pairs (L,c), where L is a vital line and c is an entry on L that contains the letter M. On one hand, there are k occurrences of M in each vital row and each vital column. For vital diagonals, there are

$$1+2+\cdots+(k-1)+k+(k-1)+\cdots+2+1=k^2$$

occurrences of M. So $N=4k^2$. On the other hand, there are $3k^2$ occurrences of M in the whole table. Note each entry belongs to exactly 1 or 4 vital lines. Hence $N\equiv 3k^2 \pmod 3$, making k a multiple of 3 and n a multiple of 9.

Problem 3. Let $P=A_1A_2\ldots A_k$ be a convex polygon in the plane. The vertices A_1, A_2, …, A_k have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

This is the hardest problem. 548 out of 602 contestants got 0 on this problem.

That $2S$ is an integer follows from the well-known *Pick's formula*, which asserts $S=I+B/2-1$, where I and B are the numbers of interior and boundary points with integral coordinates respectively.

Below we will outline the cleverest solution due to Dan Carmon, the leader of Israel. It suffices to consider the case $n=p^t$ with p prime, $t\ge 1$. By multiplying the denominator and translating, we may assume the center O is a point with integral coordinates, which we can move to the origin. We can further assume the x, y coordinates of the vertices are coprime and there exists i with x_i, y_i not both multiples of p. Then we make two claims:

(1) For $\triangle ABC$ with integral coordinates, suppose $n\mid AB^2, BC^2$ and let S be its area. Then $n\mid 2S$ if and only if $n\mid AC^2$.

(2) For those i such that x_i, y_i not both multiples of p, let Δ be twice the area of triangle $A_{i-1}A_iA_{i+1}$. Then p^t divides Δ.

For (1), note that $2S=\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$,

$$AC^2=AB^2+BC^2-2\overrightarrow{BA}\cdot\overrightarrow{BC}$$
$$\equiv -2\overrightarrow{BA}\cdot\overrightarrow{BC} \pmod n$$

and $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|^2+\left|\overrightarrow{BA}\cdot\overrightarrow{BC}\right|^2=AB^2BC^2\equiv 0 \pmod{n^2}$.

For (2), assume p^t does not divide Δ. Note O is defined by the intersection of the perpendicular bisectors, which can be written as the following system of vectors:

$$\overrightarrow{A_iA_{i+1}}\cdot\overrightarrow{A_iO}=\tfrac{1}{2}A_iA_{i+1}^2,\quad \overrightarrow{A_iA_{i-1}}\cdot\overrightarrow{A_iO}=\tfrac{1}{2}A_iA_{i-1}^2.$$

Say $\overrightarrow{A_iA_{i+1}}=(u_1,v_1)$, $\overrightarrow{A_iA_{i-1}}=(u_2,v_2)$. Using the fact that p^t does not divide $\Delta=|u_1v_2-u_2v_1|$, one can conclude that x_i, y_i are divisible by p by Cramer's rule. The rest of the solution follows by induction on the number of sides of the polygon and the two claims.

Problem 4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n)=n^2+n+1$. What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

One can begin by looking at facts like

1. $\gcd(P(n),P(n+1))=1$ for all n
2. $\gcd(P(n),P(n+2))=1$ for $n\not\equiv 2 \pmod 7$
3. $\gcd(P(n),P(n+2))=7$ for $n\equiv 2 \pmod 7$
4. $\gcd(P(n),P(n+3))=1$ for $n\not\equiv 1 \pmod 3$
5. $3\mid\gcd(P(n),P(n+3))$ for $n\equiv 1 \pmod 3$.

Assume $P(a)$, $P(a+1)$, $P(a+2)$, $P(a+3)$, $P(a+4)$ is fragrant. By 1, $P(a+2)$ is coprime to $P(a+1)$ and $P(a+3)$. Next assume $\gcd(P(a),P(a+2))>1$. By 3, $a\equiv 2 \pmod 7$. By 2, $\gcd(P(a+1),P(a+3))=1$. In order for the set to be fragrant, we must have both $\gcd(P(a),P(a+3))$ and $\gcd(P(a+1),P(a+4))$ be greater than 1. By 5, this holds only when a and $a+1\equiv 1 \pmod 3$, which is a contradiction.

For a fragrant set with 6 numbers, we can use the Chinese remainder theorem to solve the system $a\equiv 7 \pmod{19}$, $a+1\equiv 2 \pmod 7$ and $a+2\equiv 1 \pmod 3$. For example, $a=197$. By 3, $P(a+1)$ and $P(a+3)$ are divisible by 7. By 5, $P(a+2)$ and $P(a+5)$ are divisible by 3. Using $19\mid P(7)=57$ and $19\mid P(11)=133$, we can check $19\mid P(a)$ and $19\mid P(a+4)$. Then $P(a)$, $P(a+1)$, $P(a+2)$, $P(a+3)$, $P(a+4)$, $P(a+5)$ is fragrant.

Problem 5. The equation

$$(x-1)(x-2)\cdots(x-2016)$$
$$=(x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***October 21, 2016.***

Problem 491. Is there a prime number p such that both p^3+2008 and p^3+2010 are prime numbers? Provide a proof.

Problem 492. In convex quadrilateral $ADBE$, there is a point C within $\triangle ABE$ such that

$\angle EAD+\angle CAB=180^\circ=\angle EBD+\angle CBA$.

Prove that $\angle ADE=\angle BDC$.

Problem 493. For $n\geq 4$, prove that $x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ cannot be factored into a product of two polynomials with rational coefficients, both with degree greater than 1.

Problem 494. In a regular n-sided polygon, either 0 or 1 is written at each vertex. By using non-intersecting diagonals, Bob divides this polygon into triangles. Then he writes the sum of the numbers at the vertices of each of these triangles inside the triangle. Prove that Bob can choose the diagonals in such a way that the maximal and minimal numbers written in the triangles differ by at most 1.

Problem 495. The lengths of each side and diagonal of a convex polygon are rational. After all the diagonals are drawn, the interior of the polygon is partitioned into many smaller convex polygonal regions. Prove that the sides of each of these smaller convex polygons are rational numbers.

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Solutions

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Problem 486. Let $a_0=1$ and

$$a_n=\frac{\sqrt{1+a_{n-1}^2}-1}{a_{n-1}}.$$

for n=1,2,3,…. Prove that $2^{n+2}a_n>\pi$ for all positive integers n.

Solution. **Charles BURNETTE** (Graduate Student, Drexel University, Philadelphia, PA, USA), **Prithwijit DE** (HBCSE, Mumbai, India), **FONG Ho Leung** (Hoi Ping Chamber Secondary School), **Mustafa KHALIL** (Instituto Superior Tecnico, Syria), **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania), **Toshihiro SHIMIZU** (Kawasaki, Japan), **WONG Yat** and **YE Jeff York**, **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

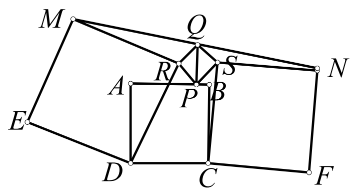
Let $a_n=\tan\theta_n$, where $0\leq\theta_n<\pi/2$. Then $a_0=1$ implies $\theta_0=\pi/4$. By the recurrence relation of a_n, we get

$$\tan\theta_n=\frac{\sec\theta_{n-1}-1}{\tan\theta_{n-1}}=\frac{1-\cos\theta_{n-1}}{\sin\theta_{n-1}}$$
$$=\frac{2\sin^2(\theta_{n-1}/2)}{2\cos(\theta_{n-1}/2)\sin(\theta_{n-1}/2)}=\tan\frac{\theta_{n-1}}{2}.$$

Then $a_n=\tan\theta_n=\tan\frac{\theta_0}{2^n}=\tan\frac{\pi}{2^{n+2}}>\frac{\pi}{2^{n+2}}$,

which is the desired inequality.

Problem 487. Let $ABCD$ and $PSQR$ be squares with point P on side AB and $AP>PB$. Let point Q be outside square $ABCD$ such that $AB\perp PQ$ and $AB=2PQ$. Let $DRME$ and $CSNF$ be squares as shown below. Prove Q is the midpoint of line segment MN.



Solution. **FONG Ho Leung** (Hoi Ping Chamber Secondary School), **Tran My LE** (Sai Gon University, Ho Chi Minh City, Vietnam) and **Duy Quan TRAN** (University of Medicine and Pharmacy, Ho Chi Minh City, Vietnam), **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Mihai STOENESCU** (Bischwiller, France), **WONG Yat** and **YE Jeff York**.

Let Q be the origin, P be $(0,-2)$ and $B=(x,-2)$. Since $AB\perp PQ$ and $PSQR$ is a square, so $S=(1,-1)$. Using $AB=2PQ=4$, we get $C=(x,-6)$. Since $CS=NS$ and $\angle CSN=90^\circ$, we get $N=(6,2-x)$.

Similarly, $R=(-1,-1)$, $D=(x-4,-6)$ and $\angle DRM=90^\circ$, so $M=(-6,x-2)$. Then the midpoint of MN is $(0,0)=Q$.

Other commended solvers: **Andrea FANCHINI** (Cantù, Italy), **Apostolos MANOLOUDIS** (4 High School of Korydallos, Piraeus, Greece) and **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India).

Problem 488. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f:\mathbb{Q}\to\{0,1\}$ satisfy $f(0)=0$, $f(1)=1$ and the condition $f(x)=f(y)$ implies $f(x)=f((x+y)/2)$. Prove that if $x\geq 1$, then $f(x)=1$.

Solution. **Jon GLIMMS.**

We first show $f(n)=1$ for n=1,2,3,… by induction. The case $n=1$ is given. For $n>1$, suppose case $n=k-1$ is true. If $f(k)=0=f(0)$, then $f(k)=f((0+k)/2)=f((1+(k-1))/2)=f(k-1)=1$, which is a contradiction.

Assume there exists rational $r>1$ such that $f(r)=0$. Suppose $r=s/t$, where s, t are coprime positive integers. Define $g:\mathbb{Q}\to\{0,1\}$ by $g(x)=1-f(w(x))$, where $w(x)=(r-[r])x+[r]$. Observe that the graph of w is a line. So $w((x+y)/2)=(w(x)+w(y))/2$.

If $g(x)=g(y)$, then $f(w(x))=f(w(y))$, which implies

$$f(w(x))=f\left(\frac{w(x)+w(y)}{2}\right)=f\left(w\left(\frac{x+y}{2}\right)\right).$$

So $g(x)=g((x+y)/2)$. Then $g(n)=1$ by induction as f above. Finally, $s>t$ implies $w(t)=(r-[r])t+[r]=s-[r]t+[r]$ is a positive integer. Then $g(t)=1-f(w(t))=0$, contradiction.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan), **WONG Yat** and **YE Jeff York**,

Problem 489. Determine all prime numbers p such that there exist positive integers m and n satisfying $p=m^2+n^2$ and m^3+n^3-4 is divisible by p.

Solution. **Prithwijit DE** (HBCSE, Mumbai, India), **Jon GLIMMS, WONG Yat** and **YE Jeff York**.

Clearly, the case $p=2$ works. For such prime $p>2$, we get $m>1$ or $n>1$. Now we have

$$(3m+3n)p-2(m^3+n^3-4)$$
$$=(m+n)^3+8$$
$$=(m+n+2)((m+n)^2-2(m+n)+4)$$
$$=(m+n+2)(p+2((m-1)(n-1)+1)).$$

Observe that $p<p+2((m-1)(n-1)+1)<p+2mn\leq p+m^2+n^2=2p$. Then p divides $m+n+2$. So $m^2+n^2\leq m+n+2$, i.e. $(m-1/2)^2+(n-1/2)^2\leq(3/2)^2$. Then

$(m,n)=(1,2)$ or $(2,1)$ and $m^3+n^3-4=5=p$. So $p=2$ and 5 are the solutions.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 490. For a parallelogram *ABCD*, it is known that $\triangle ABD$ is acute and *AD*=1. Prove that the unit circles with centers *A, B, C, D* cover *ABCD* if and only if

$$AB \le \cos\angle BAD + \sqrt{3}\sin\angle BAD.$$

Solution. **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We first show that the unit circles with centers *A, B, C, D* cover *ABCD* if and only if the circumradius *R* of $\triangle ABD$ is not greater than 1. Since $\triangle ABD$ is acute, its circumcenter *O* is inside the triangle. Then at least one of *B* or *D* is closer than (or equal to) *C* to *O*, since the region in $\triangle CDB$ that is closer to *C* than both *B* and *D* is the quadrilateral *CMO'N*, where *M* is the midpoint of *CD*, *O'* is the circumcenter of $\triangle CDB$ and *N* is the midpoint of *BC*. So for any point *P* in $\triangle ABD$, $\min\{PA,PB,PD\} \le PC$ and the maximal value of $\min\{PA,PB,PD\}$ is attained when *P*=*O*. So the unit circles with centers *A, B, C, D* cover *ABCD* is equivalent to they cover *O*, which is equivalent to $R\le 1$.

Let $\alpha=\angle BAD$, $\beta=\angle ADB$ and $\gamma=\angle DBA$. By sine law, $AB/\sin\beta=1/\sin\gamma=2R$. Then, we have

$$AB=\frac{\sin\beta}{\sin\gamma}=\frac{\sin(\alpha+\gamma)}{\sin\gamma}$$
$$=\frac{\sin\alpha\cos\gamma+\cos\alpha\sin\gamma}{\sin\gamma}$$
$$=\cos\alpha+\cot\gamma\sin\alpha.$$

Moreover, $R\le 1$ is equivalent to $1\ge 1/(2\sin\gamma)$ or $\sin\gamma\ge 1/2=\sin 30^\circ$ or $\gamma\ge 30^\circ$ or $\cot\gamma\le\sqrt{3}$. Therefore, it is equivalent to $AB\le\cos\alpha+\sqrt{3}\sin\alpha$.

Other commended solvers: **WONG Yat** and **YE Jeff York**.

Olympiad Corner

(Continued from page 1)

Problem 4. Prove that for any positive integer *n*, $2\cdot\sqrt{3}\cdot\sqrt[3]{4}\cdots\sqrt[n-1]{n}>n$.

Problem 5. Let *O* be the circumcenter of the acute triangle *ABC*. Let c_1 and c_2 be the circumcircles of triangles *ABO* and *ACO*. Let *P* and *Q* be points on c_1 and c_2 respectively, such that *OP* is a diameter of c_1 and *OQ* is a diameter of c_2. Let *T* be the intersection of the tangent to c_1 at *P* and the tangent to c_2 at *Q*. Let *D* be the second intersection of the line *AC* and the circle c_1. Prove that points *D, O* and *T* are collinear.

Problem 6. A circle is divided into arcs of equal size by *n* points ($n\ge 1$). For any positive integer *x*, let $P_n(x)$ denote the number of possibilities for coloring all those points, using colors from *x* given colors, so that any rotation of the coloring by $i\cdot 360^\circ/n$, where *i* is a positive integer less than *n*, gives a coloring that differs from the original in at least one point. Prove that the function $P_n(x)$ is a polynomial with respect to *x*.

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(Continued from page 2)

For this problem, observe we need to erase at least 2016 factors. Consider erasing all factors $x-k$ with $k\equiv 2,3 \pmod 4$ on the left and $x-k$ with $k\equiv 0,1 \pmod 4$ on the right to get the equation

$$\prod_{j=0}^{503}(x-4j-1)(x-4j-4)=\prod_{j=0}^{503}(x-4j-2)(x-4j-3)$$

There are 4 cases we have to check.

(1) For $x=1,2,\cdots,2016$, one side is 0 and the other nonzero.

(2) For $x\in(4k+1,4k+2)\cup(4k+3,4k+4)$ where $k=0,1,\ldots,503$, if $j=0,1,\ldots,503$ and $j\ne k$, then $(x-4j-1)(x-4j-4)>0$, but if $j=k$, then $(x-4k-1)(x-4k-4)<0$ so that the left side is negative. However, on the right side, each product $(x-4j-2)(x-4j-3)$ is positive, which is a contradiction.

(3) For $x<1$ or $x>2016$ or $x\in(4k,4k+1)$, where $k=0,1,\ldots,503$, dividing the left side by the right, we get

$$1=\prod_{j=0}^{503}\left(1-\frac{2}{(x-4j-2)(x-4j-3)}\right).$$

Note $(x-4j-2)(x-4j-3)>2$ for $j=0,1,\ldots,503$. Then the right side is less than 1, contradiction.

(4) For $x\in(4k+2,4k+3)$, where $k=0,1,\ldots,503$, dividing the left side by the right, we get

$$1=\frac{x-1}{x-2}\,\frac{x-2016}{x-2015}\prod_{j=1}^{503}\left(1+\frac{2}{(x-4j+1)(x-4j-2)}\right).$$

The first two factors on the right are greater than 1 and the factor in the parenthesis is greater than 1, which is a contradiction.

Problem 6. There are $n>2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hand $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if *n* is odd.

(b) Prove that Geoff can never fulfill his wish if *n* is even.

Unlike previous years, this problem 6 was not as hard as problem 3. There were 474 out of 602 contestants, who got 0 on this problem.

Take a disk containing all segments. Extend each segment to cut the boundary of the disk at points A_i, B_i.

(a) For odd *n*, go along the boundary and mark all these points 'in' and 'out' alternately. For each A_iB_i rename the 'in' point as A_i and 'out' point as B_i. Geoff can put a frog on each of the 'in' points. Let $A_iB_i\cap A_kB_k=P$. There are $n-1$ points on the open segment A_iB_i for every *i*. On the *open* arc A_iA_k, there is an <u>odd</u> number of points due to the alternate naming of the boundary points. Each of the points on *open* arc A_iA_k is a vertex of some A_xB_x, which intersects a unique point on either open segment A_iP or A_kP. So the number of points on open segments A_iP and A_kP are of opposite parity. Then the frogs started at A_i and A_k cannot meet at *P*.

(b) For even *n*, let Geoff put a frog on a vertex of a A_iB_i segment, say the frog is at A_i, which is the 'in' point and B_i is the 'out' point. As *n* is even, there will be two neighboring points labeled A_i and A_k. Let $A_iB_i\cap A_kB_k=P$. Then any other segment A_mB_m intersecting one of the open segments A_iP or A_kP must intersect the other as well. So the number of intersection points by the other segments on open segments A_iP and A_kP are the same. Then the frogs started at A_i and A_k will meet at *P*.

Solutions to Olympiad Corner, vol. 20 no. 5

(1) *Solution.* Call a position *balanced*, if the non-empty heaps can be divided into pairs, with an equal number of stones in both heaps of each pair. We show that in a balanced position, the player who moves second has a winning strategy. If there are no heaps left, the second player has already won. In the general case, suppose that the first player picks the heap H, takes n stones from it off the table, and moves a_1 stones to the first heap, a_2 stones to the second, etc. If any stones are left in H after that, the second player can then pick the heap H' that is paired with H, remove n stones from it, and move a_1 stones to the heap paired with the first heap, a_2 stones to the heap paired with the second heap, etc. On the other hand, if the first player empties the heap H, the second player does the same as in the first case, with the following exception: if the first player moved any stones to H', the second player takes this many additional stones off the table instead of moving them back to H, ensuring that the heap H' also becomes empty. In both cases, the position after the second player's move is balanced again. Since the number of stones on the table decreases with each move and the second player can always make a move, the second player eventually wins.

Finally, we show that Jüri can move into a balanced position with his first move, giving him a winning strategy. Number the heaps $1, 2, \ldots, k$ in decreasing order of size, and let the sizes of the heaps be $a_1 > a_2 > \cdots > a_k$. Jüri should pick heap 1 and move $a_2 - a_3$ stones to heap 3, $a_4 - a_5$ stones to heap 5, etc. If k is odd, Jüri should take all stones remaining in heap 1 off the table; if k is even, he should leave a_k stones in heap 1. It remains to verify that this move is possible. If $k = 1$, the game will be over after Jüri's move. For $k > 1$, $(a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{k-1} - a_k) < (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) + \cdots + (a_{k-1} - a_k) = a_1 - a_k$, which shows that after redistributing stones to odd-numbered heaps, there are still more than a_k stones left in heap 1, so the move is possible for both odd and even k. Furthermore, the strictness of the inequality ensures that some stones will be left to take off the table as required.

(2) *Solution.* Suppose a, b, c satisfy the equation. As p is positive, this implies that $|a|^b|b|^c|c|^a = p$. Clearly none of a, b, c can be zero.

Observe that $\gcd(a,b,c) = 1$. Indeed, if $d \mid a$, $d \mid b$, $d \mid c$, then the exponent of p in the canonical representation of each of the positive rational numbers $|a|^b$, $|b|^c$, $|c|^a$ is divisible by d. Hence the exponent of p in the canonical representation of the product $|a|^b|b|^c|c|^a$ is divisible by d. As this product equals to p, we get $|d| = 1$.

Consider now arbitrary prime number q different from p. Let α, β, γ be the exponents of q in the canonical representation of the positive integers $|a|, |b|, |c|$, respectively. Then $\alpha b + \beta c + \gamma a = 0$ whereby not all exponents α, β, γ are positive because $\gcd(a,b,c) = 1$. Consequently, if some of α, β, γ is positive, then there must be exactly two positive exponents among α, β, γ. W.l.o.g., assume $\alpha > 0$, $\beta > 0$, $\gamma = 0$. Then $\alpha b + \beta c = 0$, implying $\alpha|b| = \beta|c|$. Hence $|b|$ divides $\beta|c|$. As q^β divides $|b|$ while q^β is relatively prime to $|c|$, this implies $q^\beta \mid \beta$ and $q^\beta \leq \beta$ which is impossible. This means that actually $\alpha = \beta = \gamma = 0$ and $|a|, |b|, |c|$ are all powers of p.

Hence the equation rewrites to $p^{\alpha b}p^{\beta c}p^{\gamma a} = p$ where α, β, γ are now the exponents of p in the canonical representation of $|a|, |b|, |c|$, respectively. This is equivalent to $\alpha b + \beta c + \gamma a = 1$. By $\gcd(a,b,c) = 1$, one of α, β, γ must be zero, and clearly, one of the summands $\alpha b, \beta a, \gamma a$ must be positive. W.l.o.g., let $\alpha b > 0$, i.e., $\alpha > 0$ and $b > 0$. Now there are three cases.

If $\beta = 0$ and $\gamma = 0$ then $b = 1$ and $|c| = 1$. Furthermore, $\alpha b + \beta c + \gamma a = 1$ reduces to $\alpha = 1$, whence $|a| = p$. If $p > 2$ then the exponents of a and c in the original equation, b and a, are both odd, whence a and c must have the same sign to make the product $a^b b^c c^a$ positive. Both triples $(p, 1, 1)$ and $(-p, 1, -1)$ satisfy the original equation. If $p = 2$ then c^a is positive anyway, hence a must be positive. Both triples $(2, 1, 1)$ and $(2, 1, -1)$ satisfy the original equation.

If $\beta = 0$ and $\gamma > 0$ then $b = 1$. Furthermore, $\alpha b + \beta c + \gamma a = 1$ reduces to $\alpha + \gamma a = 1$, whence $a < 0$. We obtain $p^\alpha \leq \gamma p^\alpha = \gamma|a| = \alpha - 1 < \alpha$ which is impossible.

If $\beta > 0$ and $\gamma = 0$ then $|c| = 1$. Furthermore, $\alpha b + \beta c + \gamma a = 1$ reduces to $\alpha b + \beta c = 1$, which gives $c = -1$ and $\alpha p^\beta = 1 + \beta$ as the only possibility. If $p > 2$ then this leads to contradiction similar to the previous case. If $p = 2$ then $\alpha = \beta = 1$ is the only solution. This leads to triples $(2, 2, -1)$ and $(-2, 2, -1)$ which both satisfy the original equation.

(3) *Solution.* Substituting $y = -2^{x-1}$ into the original equation gives

$$f(0) = \frac{1}{2^{2^{x-1}}} f(f(x)) f(-2^{x-1}). \qquad (1)$$

So, if $f(-2^x) = 0$ for at least one x then also $f(0) = 0$. Then taking $x = 0$ and arbitrary y in the original identity gives $f(1+2y) = 0$, i.e., $f \equiv 0$.

Assume in the rest that $f(-2^x) \neq 0$ for every x. Substituting $y = -2^x$ into the original equation gives $f(-2^x) = \frac{1}{2^{2^x}} f(f(x)) f(-2^x)$. Hence, for every x,

$$f(f(x)) = 2^{2^x} \qquad (2)$$

Substituting (2) into the original equation and taking $y = 0$, we obtain $f(2^x) = f(f(x))f(0) = 2^{2^x} f(0)$ for all x, which implies

$$f(x) = 2^x f(0) \qquad (3)$$

for all positive x. On the other hand, applying (2) to (1) gives

$$f(-2^{x-1}) = \frac{2^{2^{x-1}}}{2^{2^x}} \cdot f(0) = 2^{-2^{x-1}} f(0)$$

for all x, which implies (3) also for all negative x.

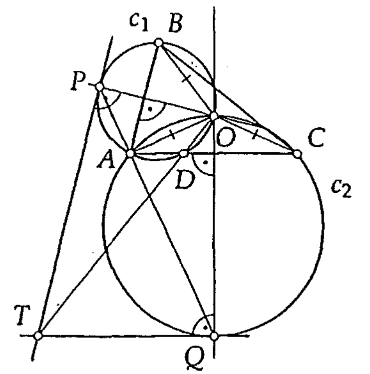
We have shown above that $f(0) = 0$ implies $f(x) = 0$ for all x. Hence we may assume that $f(0)$ is either positive or negative. By taking $x = 0$ in (2) and applying (3), we obtain $2 = 2^{2^0} = f(f(0)) = 2^{f(0)} \cdot f(0)$. Both $f(0) < 1$ and $f(0) > 1$ would lead to contradiction, hence $f(0) = 1$ and the only non-zero solution is thus $f(x) = 2^x$.

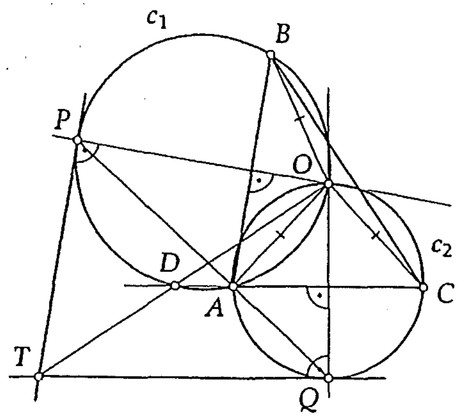
(4) *Solution.* For $2 \leq k \leq n$, the GM-HM inequality for the numbers $k, \ldots, k, 1$, with k repeated $k-2$ times, gives

$$k - 1 = \frac{(k-1)^2}{k-1} = \frac{k(k-2)+1}{k-1} \geqslant \sqrt[k-1]{k^{k-2}} = \sqrt[k-1]{\frac{k^{k-1}}{k}} = \frac{k}{\sqrt[k-1]{k}}.$$

Hence $\sqrt[k-1]{k} \geqslant \frac{k}{k-1}$ for every $k = 2, 3, \ldots, n$, with equality only for $k = 2$. Therefore

$$2 \cdot \sqrt{3} \cdot \sqrt[3]{4} \cdot \ldots \cdot \sqrt[n-1]{n} > \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n}{n-1} = n.$$





(5) *Solution.* Since $\angle OAP = \angle OAQ = 90°$, the points P, A and Q are collinear. Since $\angle OPT = \angle OQT = 90°$, $OPTQ$ is cyclic. Since $OA = OB$, the diameter OP of c_1 is perpendicular to the chord AB. Therefore PT and AB are parallel. Now $\angle TOQ = \angle TPQ = \angle TPA = \angle BAP = \angle BOP = 90° - \angle ABO$. On the other hand, equality of inscribed angles subtending the arc AO of circle c_1 gives $\angle CDO = \angle ABO$ (figures 20 and 21 show two possible situations). Therefore $\angle DOQ = 90° - \angle CDO = 90° - \angle ABO$. In summary, $\angle TOQ = \angle DOQ$, whence D, O and T are collinear.

(6) *Solution 1.* Call a colouring of the n points *permissible* if it satisfies the conditions of the problem (is not invariant under any non-full rotation). Call two colourings *equivalent* if any two points are coloured the same by the first colouring iff they are coloured the same by the second. Clearly, any two equivalent colourings use the same number of colours, and if one is permissible, so is the other. Consider an equivalence class whose colourings use exactly y colours. The number of colourings in this class that use colours from a given set of x colours is $x(x-1)\ldots(x-y+1)$. This holds also for $x < y$, the product then being zero. $P_n(x)$ is equal to the sum of those products over all equivalence classes of permissible colourings, and is therefore a polynomial with respect to x.

Solution 2. Consider all colourings of the n points with colours from among x given colours. The total number of the colourings is x^n. Let the *period* of a colouring be the least positive number d such that d/n of a full rotation gives the same colouring. $P_n(x)$ is therefore the number of colourings with period n. A standard argument gives that the period of any colouring is a divisor of n.

Let's count the number of colourings with period d. Any such colouring is determined by the colouring of the first d points, and there may be no smaller period among those points; therefore, there are $P_d(x)$ such colourings. Now we get $x^n = \sum_{d|n} P_d(x)$ and therefore $P_n(x) = x^n - \sum_{d|n,\, d<n} P_d(x)$. Since $P_1(x) = x$, induction by n now gives that $P_n(x)$ is a polynomial for any n.