Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Second Round of the *32nd Iranian Math Olympiad*.

Problem 1. A local supermarket is responsible for the distribution of 100 supply boxes. Each box is ought to contain 10 kilograms of rice and 30 eggs. It is known that a total of 1000 kilograms of rice and 3000 eggs are in these boxes, but in some of them the amount of either item is more or less than the amount required. In each step, supermarket workers can choose two arbitrary boxes and transfer any amount of rice or any number of eggs between them. At least how many steps are required so that, starting from any arbitrary initial condition, after these steps the amount of rice and the number of eggs in all these boxes are equal?

Problem 2. Square $ABCD$ is given. Points N and P are selected on sides AB and AD, respectively, such that $PN = NC$, and point Q is selected on segment AN such that $\angle NCB = \angle QPN$. Prove that $\angle BCQ = \frac{1}{2}\angle PQA$.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 29, 2016***.

For individual subscription for the next five issues for the 15-16 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Coloring Problems

Kin Y. Li

In some math competitions, there are certain combinatorial problems that are about partitioning a board (or a set) into pieces like dominos. We will look at some of these interesting problems. Often clever ways of assigning color patterns to the squares of the board allow simple solutions. Below, a <u>$m\times n$ rectangle</u> will mean a m-by-n or a n-by-m rectangle.

<u>*Example 1.*</u> A 8×8 chessboard with the the northeast and southwest corner unit squares removed is given. Is it possible to partition such a board into thirty-one dominoes (where a domino is a 1×2 rectangle)?

<u>*Solution.*</u> For such a board, we can color the unit squares alternatively in black and white, say black is color 1 and white is color 2. Then we have the following pattern.

1	2	1	2	1	2	1	
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2
	1	2	1	2	1	2	1

Each domino will cover two adjacent squares, one with color 1 and the other with color 2. If 31 dominoes can cover the board, there should be 31 squares with color 1 and 31 squares with color 2. However, in the board above there are 32 squares of color 1 and 30 squares of color 2. So the task is impossible.

<u>*Example 2.*</u> Eight 1×3 rectangles and one 1×1 square covered a 5×5 board. Prove that the 1×1 square must be over the center unit square of the board.

<u>*Solution.*</u> Let us paint the 25 unit squares of the 5×5 board with colors A, B and C as shown on the top of the next column.

A	B	C	A	B
B	C	A	B	C
C	A	B	C	A
A	B	C	A	B
B	C	A	B	C

There are 8 color A squares, 9 color B squares and 8 color C squares. Each 1×3 rectangle covers a color A, a color B and a color C square. So the 1×1 square piece must be over a color B square.

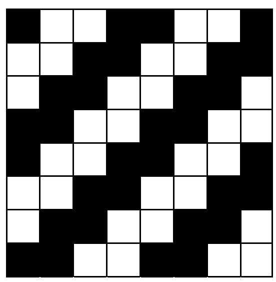
Next, we rotate the <u>*coloring*</u> of the board (not the board itself) clockwise 90˚ around the center unit square.

B	A	C	B	A
C	B	A	C	B
A	C	B	A	C
B	A	C	B	A
C	B	A	C	B

Then observe that the 1×1 square piece must still be over a color B square due to reasoning used in the top paragraph. However, the only color B square that remains color B after the 90˚ rotation is the center unit square. So the 1×1 square piece must be over the center unit square.

<u>*Example 3.*</u> Can a 8×8 board be covered by fifteen 1×4 rectangles and one 2×2 square without overlapping?

<u>*Solution.*</u> Consider the following coloring of the 8×8 board.



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In the coloring of the board, there are 32 white and 32 black squares respectively. By simple checking, we can see every 1×4 rectangle will cover 2 white and 2 black squares. The 2×2 square will cover *either* 1 black and 3 white squares *or* 3 black and 1 white squares. Assume the task is possible. Then the 16 pieces together should cover *either* 31 black and 33 white squares *or* 33 black and 31 white squares, which is a contradiction to the underlined statement above.

In coloring problems, other than assigning different colors to all the squares, sometimes assigning different numerical values for different types of squares can be useful in solving the problem. Below is one such example.

Example 4. Let m,n be integers greater than 2. Color every 1×1 square of a $m\times n$ board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then call this pair of squares a *distinct pair*. Let S be the number of distinct pairs in the $m\times n$ board. Prove that whether S is odd or even depends only on the 1×1 squares on the boundary of the board excluding the 4 corner 1×1 squares.

Solution. We first divide the 1×1 squares into three types. Type 1 squares are the four 1×1 squares at the corners of the board. Type 2 squares are the 1×1 squares on the boundary of the board, but not the type 1 squares. Type 3 squares are the remaining 1×1 squares.

Assign every white 1×1 square the value 1 and every black 1×1 square the value −1. Let the type 1 squares have values a, b, c, d respectively. Let the type 2 squares have values $x_1, x_2, \ldots, x_{2m+2n-8}$ and the type 3 squares have values $y_1, y_2, \ldots, y_{(m-2)(n-2)}$.

Next for every pair of 1×1 squares sharing a common edge, write the product of the values in the two squares on their common edge. Let H be the product of these values on all the common edges. For every type 1 square, it has two neighbor squares sharing a common edge with it. So the number in a type 1 square appears two times as factors in H. For every type 2 square, it has three neighbor squares sharing a common edge with it. So the number in a type 2 square appears three times as factors in H. For every type 3 square, it has four neighbor squares sharing a common edge with it. So the number in a type 3 square appears four times as factors in H. Hence,

$$\begin{aligned} H &= (abcd)^2(x_1x_2\cdots x_{2m+2n-8})^3(y_1y_2\cdots y_{(m-2)(n-2)})^4 \\ &= (x_1x_2\cdots x_{2m+2n-8})^3. \end{aligned}$$

If $x_1x_2\cdots x_{2m+2n-8}=1$, then $H=1$ and there are an even number of distinct pairs in the board. If $x_1x_2\cdots x_{2m+2n-8}=-1$, then $H=-1$ and there are an odd number of distinct pairs in the board. So whether S is even or odd is totally determined by the set of type 2 squares.

Next we will look at problems about coloring elements of some sets.

Example 5. There are 1004 distinct points on a plane. Connect each pair of these points and mark the midpoints of these line segments black. Prove that there are at least 2005 black points and there exists a set of 1004 distinct points generating exactly 2005 black midpoints of the line segments connecting pairs of them.

Solution. From 1004 distinct points, we can draw $k={}_{1004}C_2$ line segments connecting pairs of them. Among these, there exists a *longest* segment AB. Now the midpoints of the line segments joining A to the other 1003 points lie inside or on the circle center at A and radius ½AB. Similarly, the midpoints of the line segments joining B to the other 1003 points lie inside or on another circle center at B and radius ½AB. These two circles intersect only at the midpoint of AB. Then there are at least 2×1003−1=2005 black midpoints generated by the line segments.

To construct an example of a set of 1004 points generating exactly 2005 black midpoints, we can simply take 0, 2, 4, …, 2006 on the x-axis. Then the black midpoints generated are exactly the point at 1, 2, 3, …, 2005 of the x-axis..

Example 6. Find all ways of coloring all positive integers such that

(1) every positive integer is colored either black or white (but not both) and

(2) the sum of two numbers with distinct colors is always colored black and their product is always colored white.

Also, determine the color of the product of two white numbers.

Solution. Other than coloring all positive integers the same color, we have the following coloring satisfying conditions (1) and (2). We claim if m and n are white numbers, then mn is a white number. To see this, assume there are m, n both white, but mn is black. Let k be black. By (1), $m+k$ is black and $(m+k)n = mn+kn$ is white. On the other hand, kn is white and mn is black. So by (2), $mn+kn$ would also be black, which is a contradiction.

Next, let j be the *smallest* white positive integer. From (2) and the last paragraph, we see every sj is white, where s is any positive integer. We will prove every positive integer p that is not a multiple of j is black. Suppose $p=qj+r$, where q is a nonnegative integer and $0 < r < j$. Since j is the smallest white integer, so r is black. When $q=0$, $p=r$ is black. When $q \ge 1$, qj is white and so by (2), $p=qj+r$ is black.

Example 7. In the coordinate plane, a point (x,y) is called a *lattice point* if and only if x and y are integers. Suppose there is a convex pentagon $ABCDE$ whose vertices are lattice points and the lengths of its five sides are all integers. Prove that the perimeter of the pentagon $ABCDE$ is an even integer.

Solution. Let us color every lattice point of the coordinate plane either black or white. If $x+y$ is even, then color (x,y) white. If $x+y$ is odd, then color (x,y) black. Notice (x,y) is assigned a color different from its four *neighbors* $(x\pm1,y)$ and $(x,y\pm1)$.

Now for each of the five sides, say AB, of the pentagon $ABCDE$, let A be at (x_1, y_1) and B be at (x_2, y_2). Also let T_{AB} to be at (x_1,y_2). Then $\triangle ABT_{AB}$ is a right triangle with AB as the hypotenuse or it is a line segment (which we can consider as a degenerate right triangle).

Since each lattice point is assigned a color different from any one of its four neighbors, the polygonal path

$$AT_{AB}BT_{BC}CT_{CD}DT_{DE}ET_{EA}A$$

has even length. For positive integers a, b, c satisfying $a^2+b^2=c^2$, since $n^2\equiv n$ (mod 2), we get $a+b\equiv c$ (mod 2). It follows the perimeter of $ABCDE$ and the length of $AT_{AB}BT_{BC}CT_{CD}DT_{DE}ET_{EA}A$ are of the same parity. So the perimeter of $ABCDE$ is even.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 29, 2016.***

Problem 481. Let S={1,2,…,2016}. Determine the least positive integer n such that whenever there are n numbers in S satisfying every pair is relatively prime, then at least one of the n numbers is prime.

Problem 482. On $\triangle ABD$, C is a point on side BD with $C\neq B,D$. Let K_1 be the circumcircle of $\triangle ABC$. Line AD is tangent to K_1 at A. A circle K_2 passes through A and D and line BD is tangent to K_2 at D. Suppose K_1 and K_2 intersect at A and E with E inside $\triangle ACD$. Prove that $EB/EC=(AB/AC)^3$.

Problem 483. In the open interval (0,1), n distinct rational numbers a_i/b_i (i=1,2,…,n) are chosen, where n>1 and a_i, b_i are positive integers. Prove that the sum of the b_i's is at least $(n/2)^{3/2}$.

Problem 484. In a multiple choice test, there are four problems. For each problem, there are choices A, B and C. For any three students who took the test, there exist a problem the three students selected distinct choices. Determine the maximum number of students who took the test.

Problem 485. Let m and n be integers such that $m>n>1$, S={1,2,…,m} and $T=\{a_1, a_2, \ldots, a_n\}$ is a subset of S. It is known that every two numbers in T do not both divide any number in S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{n}.$$

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Solutions

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Problem 476. Let p be a prime number. Define sequence a_n by a_0=0, a_1=1 and $a_{k+2}=2a_{k+1}-pa_k$. If one of the terms of the sequence is −1, then determine all possible value of p.

Solution. **Jon GLIMMS** and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

Observe that $p\neq 2$ (otherwise beginning with a_2, the rest of the terms will be even, then −1 cannot appear). On one hand, using the recurrence relation, we get

$$a_{k+2}\equiv 2a_{k+1}\equiv\cdots\equiv 2^{k+1}a_1=2^{k+1} \pmod p.$$

If $a_m=-1$ for some $m\geq 2$, then letting $k=m-2$, we get

$$-1=a_m\equiv 2^{m-1} \pmod p. \qquad (*)$$

On the other hand, using the recurrence relation again, we also have

$$a_{k+2}\equiv 2a_{k+1}-a_k \pmod{p-1},$$

which implies $a_{k+2}-a_{k+1}\equiv a_{k+1}-a_k\equiv\cdots\equiv a_1-a_0=1 \pmod{p-1}$. Then

$$-1=a_m\equiv m+a_0=m \pmod{p-1},$$

which implies $p-1$ divides $m+1$. By Fermat's little theorem and (*), we get

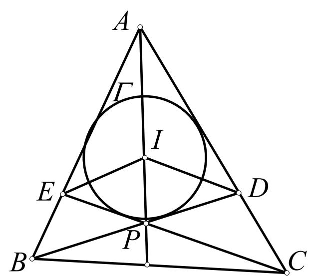
$$1\equiv 2^{m+1}\equiv 4\cdot 2^{m-1}\equiv -4 \pmod p.$$

Then p=5. Finally, if p=5, then $a_3=-1$.

Problem 477. In $\triangle ABC$, points D, E are on sides AC, AB respectively. Lines BD and CE intersect at a point P on the bisector of $\angle BAC$.

Prove that quadrilateral $ADPE$ has an inscribed circle if and only if $AB=AC$.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Jafet Alejandro Baca OBANDO** (IDEAS High School, Nicaragua) and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Suppose $ADPE$ has an inscribed circle Γ. Since the center of Γ is on the bisector of $\angle BAC$, the center is on line AP. Similarly, AP also bisects $\angle DPE$, so $\angle APE=\angle APD$. It also follows that $\angle APB=\angle APC$, since $\angle EPB=\angle DPC$. By ASA, we get $\triangle APB\cong\triangle APC$ with AP common. Then $AB=AC$.

Conversely, if $AB=AC$, then $\triangle ABC$ is symmetric with respect to AP. Thus, lines BP and CP (hence also D and E) are symmetric with respect to AP. By symmetry, the bisectors of $\angle ADP$ and $\angle AEP$ meet at a point I on AP. Then the distances from I to lines EA, EP, DP, DA are the same. So $ADPE$ has an inscribed circle with center I.

Other commended solvers: **Mark LAU Tin Wai** (Pui Ching Middle School), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 478. Let a and b be a pair of coprime positive integers of opposite parity. If a set S satisfies the following conditions:

(1) $a, b\in S$;
(2) if $x,y,z\in S$, then $x+y+z\in S$,

then prove that every positive integer greater than $2ab$ belongs to S.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

Without loss of generality, we assume that a is odd and b is even. Let $n>2ab$. Since a and b are coprime, the equation $ax\equiv n \pmod b$ has a solution satisfying $0\leq x<b$. Then $y=(n-ax)/b$ is a positive integer. Now

$$a=\frac{2ab-ab}{b}<\frac{n-ax}{b}=y\leq\frac{2ab}{b}=2a.$$

Let $x'=x+b$, $y'=y-a$, then x', y' are positive and $ax'+by'=n$. Observe $x+y$ and $x'+y'=x+y+b-a$ are of opposite parity. So we may assume $x+y$ is odd (otherwise take $x'+y'$). Then $x+y\geq 3$ and by (1) and (2),

$$n=a+\cdots+a+b+\cdots+b\in S,$$

where a appeared x times and b appeared y times.

Other commended solvers: **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5) and **Mark LAU Tin Wai** (Pui Ching Middle School).

Problem 479. Prove that there exists infinitely many positive integers k such that for every positive integer n, the number $k2^n+1$ is composite.

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

By the Chinese remainder theorem, there exist infinitely many positive integers k such that

$$k \equiv 1 \pmod 3,$$
$$k \equiv 1 \pmod 5,$$
$$k \equiv 3 \pmod 7,$$
$$k \equiv 10 \pmod{13},$$
$$k \equiv 1 \pmod{17},$$
$$k \equiv -1 \pmod{241}.$$

If $n\equiv 1 \pmod 2$, then $k2^n+1\equiv 2+1\equiv 0 \pmod 3$. Otherwise $2|n$. If $n\equiv 2 \pmod 4$, then $k2^n+1\equiv 2^2+1\equiv 0 \pmod 5$. Otherwise $4|n$. If $n\equiv 4 \pmod 8$, then $k2^n+1 \equiv 2^4+1\equiv 0 \pmod{17}$. Otherwise $8|n$. Then we have three cases:

<u>*Case 1: $n\equiv 8 \pmod{24}$.*</u> By Fermat's little theorem, $2^{24} = (2^{12})^2 \equiv 1 \pmod{13}$. So $2^n = 2^{8+24m} \equiv 256 \equiv -4 \pmod{13}$ and $k2^n+1 \equiv 10(-4)+1 \equiv 0 \pmod{13}$.

<u>*Case 2: $n\equiv 16 \pmod{24}$.*</u> Since $2^{24} = (2^3)^8 \equiv 1 \pmod 7$, we have $2^n = 2^{16+24m} \equiv 2^{1+3(5+8m)} \equiv 2 \pmod 7$ and $k2^n+1 \equiv 3\cdot 2+1 \equiv 0 \pmod 7$.

<u>*Case 3: $n\equiv 0 \pmod{24}$.*</u> Since $2^{24} = (2^8)^3 \equiv 15^3 \equiv 225\cdot 15\equiv -16\cdot 15\equiv 1 \pmod{241}$. So $2^n = 2^{24m} \equiv 1 \pmod{241}$ and Then $k2^n+1 \equiv -1+1 = 0 \pmod{241}$.

Comment: We may wonder why modulo 3, 5, 7, 13, 17, 241 work. It may be that in dealing with $n\equiv 8, 16, 0 \pmod{24}$, we want $2^{24}\equiv 1 \pmod p$ for some useful primes p. Then we notice

$$2^{24}-1 = (2^3-1)(2^3+1)(2^6+1)(2^{12}+1) = 7\cdot 3^2\cdot 5\cdot 13\cdot 17\cdot 241.$$

Other commended solvers: **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Prishtina Math Gymnasium Problem Solving Group** (Republic of Kosova), **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 480. Let m, n be integers with $n > m > 0$. Prove that if $0 < x < \pi/2$, then

$$2|\sin^n x-\cos^n x| \le 3|\sin^m x-\cos^m x|.$$

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5).

If $x=\pi/4$, both sides are 0. Since the inequality for x and $\pi/2-x$ are the same, we only need to consider $0 < x < \pi/4$. Let $k\ge 0$. Define $a_k=\cos^k x-\sin^k x$. We have $a_k\ge 0$. For $k\ge 2$, we have

$$\begin{aligned} a_k &= (\cos^k x-\sin^k x)(\cos^2 x+\sin^2 x) \\ &= a_{k+2}+\sin^2 x\cos^2 x\, a_{k-2} \\ &\ge a_{k+2}. \end{aligned}$$

Let $m \ge 2$. For the case $n-m = 2,4,6,\ldots$, we have $3a_m \ge 2a_m \ge 2a_n$. Next, for the case $n-m = 1,3,5,\ldots$, observe that

$$(\cos x+\sin x)a_m=a_{m+1}+\sin x\cos x\, a_{m-1}.$$

Using this, we have

$$\begin{aligned} &3a_m \ge 2a_{m+1} \\ \Leftrightarrow\ &3a_m \ge 2[(\cos x+\sin x)a_m - \sin x\cos x\, a_{m-1}] \\ \Leftrightarrow\ &[3-2\sqrt{2}\sin(x+\tfrac{\pi}{4})]a_m \ge -2\sin x\cos x\, a_{m-1}, \end{aligned}$$

which is true as the left side is positive and the right side is negative. Then $3a_m \ge 2a_{m+1} \ge 2a_n$.

Finally, for the case $m=1$, we get $3a_1\ge 2a_2$ from $3 > 2\sqrt{2} \ge 2(\cos x + \sin x) = 2a_2/a_1$. Then $3a_1 \ge 2a_2 \ge 2a_n$ for $n = 2, 4, 6, \ldots$. Also, we get $3a_1\ge 2a_3$ from $3 \ge 2+\sin 2x = 2a_3/a_1$. Then $3a_1 \ge 2a_3 \ge 2a_n$ for $n = 3, 5, 7, \ldots$.

Other commended solvers: **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Olympiad Corner

(Continued from page 1)

Problem 3. Let x, y and z be nonnegative real numbers. Knowing that $2(xy+yz+zx) = x^2+y^2+z^2$, prove

$$\frac{x+y+z}{3} \ge \sqrt[3]{2xyz}.$$

Problem 4. Find all of the solutions of the following equation in natural numbers:

$$n^{n^n} = m^m.$$

Problem 5. A non-empty set S of positive real numbers is called **powerful** if for any two distinct elements of it like a and b, at least one of the numbers a^b or b^a is an element of S.

a) Present an example of a powerful set having four elements.

b) Prove that a finite powerful set cannot have more than four elements.

Problem 6. In the **Majestic Mystery Club (MMC)**, members are divided into several groups, and groupings change by the end of each week in the following manner: in each group, a member is selected as king; all of the kings leave their respective groups and form a new group. If a group has only one member, that member goes to the new group and his former group is deleted. Suppose that MMC has n members and at the beginning all of them form a single group. Prove that there comes a week for which thereafter each group will have at most $1+\sqrt{2n}$ members.

Coloring Problems

(Continued from page 2)

<u>*Example 8.*</u> Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are divided into two groups, each having at least one number. Prove that there always exists a three term arithmetic progression (AP in short) in one of the two groups.

<u>*Solution.*</u> Assume no three term AP is in any of the two groups. Color numbers in one group red and the other group blue. Since 5/2>2, among 1, 3, 5, 7, 9, there exist three of them assigned the same color, say they are red. By assumption, they are not the terms of an AP. Below are the possibilities of these red numbers: {1,3,7}, {1,3,9}, {1,5,7}, {1,7,9}, {3,5,9} or {3,7,9}.

If 1,3,7 are red, then as 1,2,3 and 1,4,7 and 3,5,7 are AP, so 2, 4, 5 are blue. As 4,5,6 and 2,5,8 are AP, so 6, 8 are red. So 6,7,8 is a red AP, contradiction.

If 1,3,9 are red, then as 1,2,3 and 1,5,9 and 3,6,9 are AP, so 2, 5, 6 are blue. As 4,5,6 and 5,6,7 an AP, so 4,7 are red. Then 1,4,7 is a red AP, contradiction.

If 1,5,7 are red, then as 1,3,5 and 5,6,7 and 1,5,9 are AP, so 3,6,9 are blue. Then 3, 6, 9 is a blue AP, contradiction.

If 1,7,9 are red, then as 1,4,7 and 1,5,9 and 7,8,9 are AP, so 4, 5, 8 are blue. As 3,4,5 and 4,5,6 an AP, so 3,6 are red. Then 3,6,9 is a red AP, contradiction.

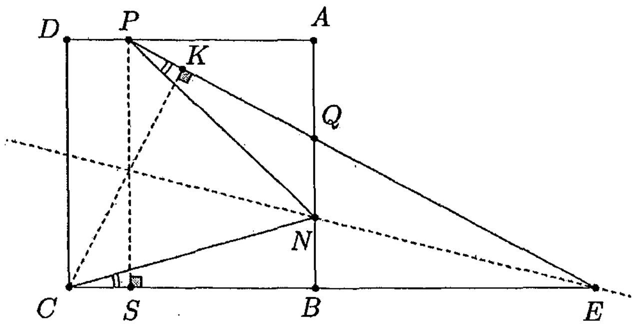
If 3,5,9 are red, then as 1,5,9 and 3,4,5 and 5,7,9 are AP, so 1,4,7 are blue. Then 1,4,7 a blue AP, contradiction.

If 3,7,9 are red, then as 3,5,7 and 3,6,9 and 7,8,9 are AP, so 5, 6, 8 are blue. As 2,5,8 and 4,5,6 are AP, so 2, 4 are red. So 2,3,4 is a red AP, contradiction.

Solutions to Olympiad Corner, vol. 20 no.3

1 . The answer is 99. First consider the initial condition that all of eggs are in just one of boxes. In each step, we can transfer eggs to at most one new box and so we need at least 99 steps. We claim that 99 steps is always enough. For this end, we call a box containing exactly 30 eggs and 10 kilograms of rice a *good box*, and a box which is not good a *bad box*! In each step, consider one of bad boxes containing the most number of eggs and one of bad boxes containing the most amount of rice. If these two boxes were the same, consider another arbitrary bad box (note that if all other boxes were good, this box also must be good and there is nothing to prove). Evidently, we have at least 30 number of eggs and 10 kilograms of rice in these two boxes. So by transfering eggs and rice between them, we can make one of them a good box. Therefore, after 99 steps we have at least 99 good boxes and so the last box is also good and we are done.

2 . Let E be the intersection point of PQ and BC. According to the problem assumption, $PN = NC$ and so $\angle NPC = \angle PCN$. On the other hand, we know $\angle QPN = \angle NCB$. From these we conclude that EPC is an isosceles triangle. Therefore, its altitudes PS and CK have equal length. So $CK = PS = AB = BC$ and therefore right-angled trinagles QBC and QKC are congruent. So QC is the bisector of angles $\angle KCB$ and $\angle KQB$. Hence, $\angle BCQ = \frac{1}{2}\angle KCB$ (*).



On the other hand, since $\angle QBC + \angle QKC = 90° + 90° = 180°$, we get that $QBCK$ is a cyclic quadrilateral which implies $\angle BCK = \angle AQP$. This together with (*) completes the proof.

3 . There is no loss of generality in assuming that $x \geq y \geq z$. We have

$$\begin{aligned} & x^2 + y^2 + z^2 = 2(xy + yz + zx) \\ \Rightarrow\ & x^2 + x(-2y - 2z) + y^2 + z^2 - 2yz = 0 \\ \Rightarrow\ & x = (y+z) \pm \sqrt{(y+z)^2 - y^2 - z^2 + 2yz} = (y+z) \pm 2\sqrt{yz} = (\sqrt{y} \pm \sqrt{z})^2 \\ \Rightarrow\ & \sqrt{x} = \sqrt{y} \pm \sqrt{z} \end{aligned}$$

Since $y, z \leq x$ the case $\sqrt{x} = \sqrt{y} - \sqrt{z}$ is not admissibles and so we get $\sqrt{x} = \sqrt{y} + \sqrt{z}$ or equivalently $x = y + z + 2\sqrt{yz}$. After substituting this equality in the statement of the problem, we deduce

$$\frac{y + z + 2\sqrt{yz} + y + z}{3} \geqslant \sqrt[3]{2(y + z + 2\sqrt{yz})yz}$$

If $y = 0$, the assertion is trivial. Therefore, we assume $y \neq 0$. Let we define $t = \frac{z}{y}$. Now we must prove

$$\frac{2t + 2\sqrt{t} + 2}{3} \geqslant \sqrt[3]{2(t + 1 + 2\sqrt{t})t}$$

Which is a consequence of AM-GM inequality.

$$\frac{2t + 2\sqrt{2t} + 2}{3} = \frac{(\sqrt{t}+1) + (\sqrt{t}+1) + 2t}{3} \geq \sqrt[3]{2(t + 1 + 2\sqrt{t})t}$$

4 . We start with a lemma.

Lemma 1. *Let n be a positive integer and p, q some positive rational numbers. If $n^p = q$, then q is itself an integer.*

Proof. Suppose $p = \frac{a}{b}$ and $q = \frac{c}{d}$ where $a, b, c, d \in \mathbb{N}$. We have

$$n^p = q \Rightarrow n^{\frac{a}{b}} = \frac{c}{d} \Rightarrow n^a = \left(\frac{c}{d}\right)^b = \frac{c^b}{d^b} \Rightarrow d^b | c^b \Rightarrow d | c \Rightarrow q \in \mathbb{N}$$

□

Now for the main problem, note that if $n = 1$, then $m = n = 1$ and this is a solution for the equation. So we may assume thagt $n > 1$. Let $r = \log_n m$, $(m = n^r)$. We have

$$n^{n^n} = m^m = (n^r)^{n^r} = n^{rn^r}$$

Since $n > 1$, we must have

$$n^n = rn^r \Rightarrow r = n^{n-r}$$

On the other hand, since $n^{n^n} = m^m$, we get $n^n = m \log_n m = r$ or $r = \frac{n^n}{m} \in \mathbb{Q}$. According to the lemma $n - r$ and r playing the role of p and q, respectively, we get r is an integer. Now if $r < n$, then $n^{n-r} \geq n^1 > r = n^{n-r}$, which is impossible. And if $r > n$, then $n^{n-r} < 1 \leq r = n^{n-r}$ which is again impossible.
So we must have $n = r$. Hence, $n = r = n^{n-r} = 1$. This contradicts with the assumption $n > 1$ and consequently, the only solution is $m = n = 1$.

5 . a) $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}\}$ is an example of a powerful set with four elements. (Part b shows that this is the unique powerful set with four elements.)

b) First we prove a lemma.

Lemma 1. *A finite powerful set S can not have an element greater than one and an element less than one.*

Proof. Suppose by contradiction that there exist such elements. Now let a be the least element of S and b the least element of S which is greater than 1. By assumption, we must have $a < 1$. Now $a < 1 < b$ and so

- $a^b < a^1 = 1$, but a was the least element of S. Thus, $a^b \notin S$.
- $1 < b^a < b^1 = b$, but b was the least element of S greater than one. Therefore, $b^a \notin S$.

So our assumption leads to a contradiction proving the lemma. □

According to this lemma, if S is a finite powerful set, all of its elements are in $[1, \infty)$ or in $(0, 1]$.
Firstly, suppose that S is a powerful set with $n > 3$ number of elements in $[1, \infty)$ ($S = \{1 = a_1 < a_2 < \cdots < a_n\}$). Note that we can assume that $a_1 = 1$, because if $1 \notin S$, we can add it to S to get a powerful set with more elements. For $i \geq 2$, $a_n^{a_i} > a_n$ and so $a_i^{a_n}$ must be in S. We have

$$a_1 < a_2 < a_2^{a_n} < a_3^{a_n} < \cdots < a_{n-1}^{a_n}$$

So for $2 \leq i \leq n-1$, $a_i^{a_n} = a_{i+1}$. Now for $a_2 < a_{n-1}$ $(n > 3)$ we have

$$a_2 < a_2^{a_{n-1}} < a_2^{a_n} = a_3 \Rightarrow a_2^{a_{n-1}} \notin S$$

$$a_{n-1} < a_{n-1}^{a_2} < a_{n-1}^{a_n} = a_n \Rightarrow {a_{n-1}}^{a_2} \notin S$$

but this contradicts, because S was a powerful set.

Now suppose that S is a powerful set with $n > 4$ elements in $(0, 1]$. Let $S = \{a_1 < a_2 < \cdots < a_n = 1\}$. Again we may assume $1 \in S$. Similar to the previous part, for each $1 \leq i \leq n-2$, $a_{n-1} < a_{n-1}^{a_i} < 1$. So $a_n^{a_i} \notin S$, and consequently $a_i^{a_n} \in S$. We have

$$a_1 < a_1^{a_{n-1}} < a_2^{a_{n-1}} < \cdots < a_{n-2}^{a_{n-1}} < 1$$

So we get $a_i^{a_{n-1}} = a_{i+1}$ for each $2 \leq i \leq n-2$. Now if we denote a_{n-1} by a, we get

$$a_{n-2} = a^{\frac{1}{a}}, a_{n-3} = a^{\frac{1}{a^2}}, \ldots$$

Now by looking at a_{n-1} and a_{n-2}, we conclude

$$a_{n-1} = a_{n-2}^{a_{n-1}} < a_{n-1}^{a_{n-2}} < 1 \Rightarrow a_{n-1}^{a_{n-2}} \notin S$$

This implies $a_{n-2}^{a_{n-1}} \in S$. Since $a_{n-2}^{a_{n-1}} > a_{n-2}^{a_n} = a_{n-1}$, we get $a_{n-2}^{a_{n-1}} = a_n$. So

$$a_{n-2}^{a_{n-1}} = \left(a^{1/a^2}\right)^{(a^{1/a})} = a \implies a^{a^{(1/a)-2}} = a \Rightarrow a^{\frac{1}{a}-2} = 1$$

But $a \neq 1$ and so $a = \frac{1}{2}$. Therefore, $a_{n-1} = \frac{1}{2}$, $a_{n-2} = \frac{1}{4}$ and $a_{n-3} = \frac{1}{16}$. Now since $n > 4$, $a_{n-4} = \frac{1}{256} \in S$ but it is easy to see none of $a_{n-3}^{a_{n-4}}$ and $a_{n-4}^{a_{n-3}}$ is not in S.
So there is no powerful set with more than 4 elements.

6 . First of all note that if we have k groups with number of members $1, 2, \ldots, k$ after a week we have again k groups with $1, 2, \ldots, k$ number of members.

Lemma 1. *If $n = \binom{k}{2}$ for some positive integer k, after $\binom{k}{2}$ weeks, there would be $k-1$ groups such that for each $1 \leq i \leq k-1$ there is a group with i members.*

Proof. The proof for $n = 2, 3$ is trivial. Suppose that the assertion is true for each integer $1 \leq k \leq m$ and now we have a group with $\binom{m+1}{2}$ members. By induction after $\binom{m}{2}$ weeks, we have $m-1$ groups with $1, 2, \ldots, m-1$ members. In such situation, the number of members in the first group will be $\binom{m+1}{2} - \binom{m}{2} = m$. Therefore, the assertion is proved. □

Now Suppose that we have $n = \binom{k}{2} + t \leq \binom{k+1}{2}$. We want to prove that after some weeks the number of members in the largest group is k. For this reason, we add $\binom{k+1}{2} - \binom{k}{2} - t$ fake members to this group to get a group with $\binom{k+1}{2}$ members. Referring to the lemma, after some weeks, for each $1 \leq i \leq k$ there always exists a group with i members. We suppose that if there is at least one real member in a group, the fake mambers cannot became king. So the number of real members in each group is at most k. This implies the assertion

$$\binom{k}{2} < n \leq \binom{k+1}{2} \Rightarrow k^2 - k + \frac{1}{4} = (k - \frac{1}{2})^2 < 2n \Rightarrow k \leq \sqrt{2n} + \frac{1}{2} < \sqrt{2n} + 1$$