Mathematical Excalibur

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Olympiad Corner

The following are the problems of the 2008 IMO held at Madrid in July.

Problem 1. An acute-angled triangle ABC has orthocenter H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with the centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Problem 2. (a) Prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\ge 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Problem 3. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor which is greater than $2n+\sqrt{2n}$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 31, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometric Transformations II

Kin Y. Li

Below the vector from X to Y will be denoted as $\boldsymbol{XY}$. The notation $\measuredangle ABC = \alpha$ means the ray BA after rotated an angle $|\alpha|$ (anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$) will coincide with the ray BC.

On a plane, a *translation* by a vector v (denoted as T(v)) moves every point X to a point Y such that $\boldsymbol{XY}$ = v. On the complex plane $\mathbb{C}$, if the vector v corresponds to the vector from 0 to v, then T(v) has the same effect as the function $f:\mathbb{C}\to\mathbb{C}$ given by $f(w)= w+v$.

A *homothety* about a center C and ratio r (denoted as $H(C,r)$) moves every point X to a point Y such that $\boldsymbol{CY} = r\,\boldsymbol{CX}$. If C corresponds to the complex number c in $\mathbb{C}$, then $H(C,r)$ has the same effect as $f(w) = r(w - c)+ c = rw + (1-r)c$.

A *rotation* about a center C by angle α (denoted as $R(C,\alpha)$) moves every point X to a point Y such that $CX = CY$ and $\measuredangle XCY = \alpha$. In $\mathbb{C}$, if C corresponds to the complex number c, then $R(C,\alpha)$ has the same effect as $f(w) = e^{i\alpha}(w - c) + c = e^{i\alpha}w + (1- e^{i\alpha})c$.

A *reflection* across a line ℓ (denoted as $S(\ell)$) moves every point X to a point Y such that the line ℓ is the perpendicular bisector of segment XY. In $\mathbb{C}$, let $S(\ell)$ send 0 to b. If $b = 0$ and ℓ is the line through 0 and $e^{i\theta/2}$, then $S(\ell)$ has the same effect as $f(w)=e^{i\theta}\overline{w}$. If $b \neq 0$, then let $b = |b|\, e^{i\beta}$, $e^{i\theta} = -e^{2i\beta}$ and L be the vertical line through $|b|/2$. In $\mathbb{C}$, $S(L)$ sends w to $|b|-\overline{w}$. Using that, $S(\ell)$ is

$$f(w)=e^{i\beta}(|b|-\overline{we^{-i\beta}})=e^{i\theta}\overline{w}+b.$$

We have the following useful facts:

Fact 1. If $\ell_1 \parallel \ell_2$, then

$$S(\ell_2)\circ S(\ell_1) = T(2\boldsymbol{A_1A_2}),$$

where A_1 is on ℓ_1 and A_2 is on ℓ_2 such that the length of $\boldsymbol{A_1A_2}$ is the distance d from ℓ_1 to ℓ_2.

(*Reason*: Say ℓ_1, ℓ_2 are vertical lines through $A_1 = 0$, $A_2 = d$. Then $S(\ell_1)$, $S(\ell_2)$ are $f_1(w)=-\overline{w}$ and $f_2(w)=-\overline{w}+2d$. So $S(\ell_2)\circ S(\ell_1)$ is

$$f_2(f_1(w)) = -\left(\overline{-\overline{w}}\right)+2d = w+2d,$$

which is $T(2\boldsymbol{A_1A_2})$.)

Fact 2. If $\ell_1 \nparallel \ell_2$, then

$$S(\ell_2)\circ S(\ell_1) = R(O,\alpha),$$

where ℓ_1 intersects ℓ_2 at O and α is twice the angle from ℓ_1 to ℓ_2 in the anticlockwise direction.

(*Reason:* Say O is the origin, ℓ_1 is the x-axis. Then $S(\ell_1)$ and $S(\ell_2)$ are

$$f_1(w)=\overline{w} \text{ and } f_2(w)=e^{i\alpha}\overline{w},$$

so $S(\ell_2)\circ S(\ell_1)$ is $f_2(f_1(w)) = e^{i\alpha}w$, which is $R(O,\alpha)$.)

Fact 3. If $\alpha+\beta$ is not a multiple of 360°, then

$$R(O_2, \beta)\circ R(O_1, \alpha) = R(O, \alpha+\beta),$$

where $\measuredangle OO_1O_2 = \alpha/2$, $\measuredangle O_1O_2O = \beta/2$. If $\alpha+\beta$ is a multiple of 360°, then

$$R(O_2, \beta)\circ R(O_1,\alpha) = T(\boldsymbol{O_1O_3}),$$

where $R(O_2, \beta)$ sends O_1 to O_3.

(*Reason*: Say O_1 is 0, O_2 is -1. Then $R(O_1, \alpha)$, $R(O_2, \beta)$ are $f_1(w) = e^{i\alpha}w$, $f_2(w) = e^{i\beta}w+(e^{i\beta}-1)$, so $f_2(f_1(w)) = e^{i(\alpha+\beta)}w + (e^{i\beta}-1)$. If $e^{i(\alpha+\beta)} \neq 1$, this is a rotation about $c' = (e^{i\beta}-1)/(1-e^{i(\alpha+\beta)})$ by angle $\alpha+\beta$. We have

$$c'=\frac{\sin(\beta/2)}{\sin((\alpha+\beta)/2)}e^{i(\pi-\alpha/2)},$$

$$c'-1=\frac{\sin(\alpha/2)}{\sin((\alpha+\beta)/2)}e^{i\beta/2}.$$

If $e^{i(\alpha+\beta)} = 1$, this is a translation by $e^{i\beta}-1 = f_2(0)$.)

Fact 4. If O_1, O_2, O_3 are noncollinear, $\alpha_1, \alpha_2, \alpha_3 > 0$, $\alpha_1+\alpha_2+\alpha_3=360°$ and

$$R(O_3,\alpha_3)\circ R(O_2,\alpha_2)\circ R(O_1, \alpha_1) = I,$$

where I is the identity transformation, then $\measuredangle O_3O_1O_2 = \alpha_1/2$, $\measuredangle O_1O_2O_3 = \alpha_2/2$ and $\measuredangle O_2O_3O_1 = \alpha_3/2$.

(This is just the case $\alpha_3=360°-(\alpha_1+\alpha_2)$ of fact 3.)

Fact 5. Let $O_1 \neq O_2$. For $r_1 r_2 \neq 1$,

$$H(O_2,r_2) \circ H(O_1,r_1) = H(O,r_1r_2)$$

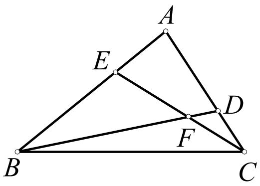
for some O on line O_1O_2. For $r_1r_2 = 1$,

$$H(O_2,r_2) \circ H(O_1,r_1) = T((1-r_2)\mathbf{O_1O_2}).$$

(*Reason*: Say O_1 is 0, O_2 is c. Then $H(O_1,r_1)$, $H(O_2,r_2)$ are $f_1(w) = r_1w$, $f_2(w) = r_2w+(1-r_2)c$, so $f_2(f_1(w)) = r_1r_2w+(1-r_2)c$. For $r_1r_2 \neq 1$, this is a homothety about $c' = (1-r_2)c/(1-r_1r_2)$ and ratio r_1r_2. For $r_1r_2 = 1$, this is a translation by $(1-r_2)c$.)

Next we will present some examples.

Example 1. In ΔABC, let E be onside AB such that $AE{:}EB$=1:2 and D be on side AC such that $AD{:}DC$ = 2:1. Let F be the intersection of BD and CE. Determine $FD{:}FB$ and $FE{:}FC$.



Solution. We have $H(E, -1/2)$ sends B to A and $H(C, 1/3)$ sends A to D. Since $(1/3)\times(-1/2) \neq 1$, by fact 5,

$$H(C, 1/3) \circ H(E, -1/2) = H(O, -1/6),$$

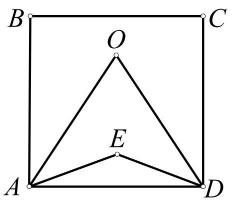
where the center O is on line CE. However, the composition on the left side sends B to D. So O is also on line BD. Hence, O must be F. Then we have FD: FB = OD: OB =1:6.

Similarly, we have

$$H(B, 2/3) \circ H(D, -2) = H(F, -4/3)$$

sends C to E, so $FE{:}FC$ = 4:3.

Example 2. Let E be inside square $ABCD$ such that $\angle EAD = \angle EDA = 15°$. Show that ΔEBC is equilateral.

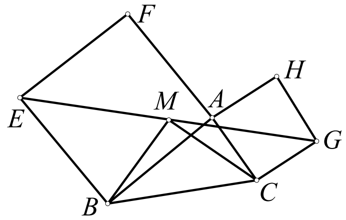


Solution. Let O be inside the square such that ΔADO is equilateral. Then $R(D, 30°)$ sends C to O and $R(A, 30°)$ sends O to B. Since $\measuredangle EDA = 15° = \measuredangle DAE$, by fact 3,

$$R(A, 30°) \circ R(D, 30°) = R(E, 60°),$$

So $R(E, 60°)$ sends C to B. Therefore, ΔEBC is equilateral.

Example 3. Let $ABEF$ and $ACGH$ be squares outside ΔABC. Let M be the midpoint of EG. Show that $MB = MC$ and $MB \perp MC$.

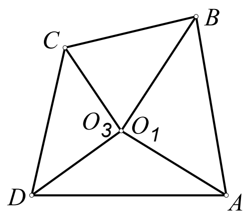


Solution. Since $GC = AC$ and $\measuredangle GCA = 90°$, so $R(C,90°)$ sends G to A. Also, $R(B, 90°)$ sends A to E. Then $R(B, 90°) \circ R(C,90°)$ sends G to E. By fact 3,

$$R(B, 90°) \circ R(C, 90°) = R(O, 180°),$$

where O satisfies $\measuredangle OCB = 45°$ and $\measuredangle CBO = 45°$. Since the composition on the left side sends G to E, O must be M. Now $\measuredangle BOC = 90°$. So $MB \perp MC$.

Example 4. On the edges of a convex quadrilateral $ABCD$, construct the isosceles right triangles ABO_1, BCO_2, CDO_3, DAO_4 with right angles at O_1, O_2, O_3, O_4 overlapping with the interior of the quadrilateral. Prove that if $O_1 = O_3$, then $O_2 = O_4$.



Solution. Now $R(O_1, 90°)$ sends A to B, $R(O_2, 90°)$ sends B to C, $R(O_3, 90°)$ sends C to D and $R(O_4, 90°)$ sends D to A. By fact 3,

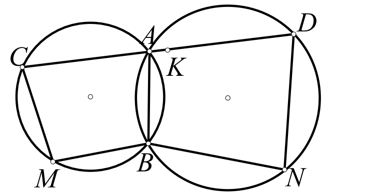
$$R(O_2, 90°) \circ R(O_1, 90°) = R(O, 180°),$$

where O satisfies $\angle OO_1O_2 = 45°$ and $\angle O_1O_2O = 45°$ (so $\angle O_2OO_1 = 90°$). Now the composition on the left side sends A to C, which implies O must be the midpoint of AC. Similarly, we have

$$R(O_4, 90°) \circ R(O_3, 90°) = R(O, 180°).$$

By fact 3, $\measuredangle O_4OO_3 = 90°$ and $\measuredangle OO_3O_4 = 45° = \measuredangle O_3O_4O$. Hence, $R(O, 90°)$ sends O_4O_2 to O_3O_1. Therefore, if $O_1 = O_3$, then $O_2 = O_4$.

Example 4. (*1999-2000 Iranian Math Olympiad, Round 2*) Two circles intersect in points A and B. A line ℓ that contains the point A intersects again the circles in the points C and D, respectively. Let M, N be the midpoints of the arcs BC and BD, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN = 90°$.



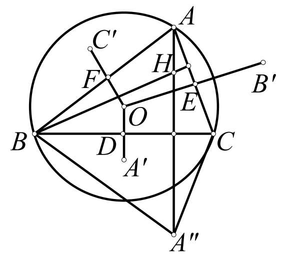
Solution. Since $\angle CAB + \angle BAD = 180°$, it follows that $\angle BMC + \angle DNB = 180°$.

Now $R(M, \measuredangle BMC)$ sends B to C, $R(K, 180°)$ sends C to D and $R(N, \measuredangle DNB)$ sends D to B. However, by fact 3,

$$R(N,\measuredangle DNB) \circ R(K,180°) \circ R(M,\measuredangle BMC)$$

is a translation and since it sends B to B, it must be the identity transformation I. By fact 4, $\measuredangle MKN = 90°$.

Example 6. Let H be the orthocenter of ΔABC and lie inside it. Let A', B', C' be the circumcenters of ΔBHC, ΔCHA, ΔAHB respectively. Show that AA', BB', CC' are concurrent and identify the point of concurrency.



Solution. For ΔABC, let O be its circumcenter and G be its centroid. Let the reflection across line BC sends A to A''. Then $\angle BAC = \angle BA''C$. Now

$\angle BHC$
$= \angle ABH + \angle BAC + \angle ACH$
$= (90° - \angle BAC) + \angle BAC + (90° - \angle BAC)$
$= 180° - \angle BA''C$.

So A'' is on the circumcircle of ΔHBC.

Now the reflection across line BC sends O to A', the reflection across line CA sends O to B' and the reflection across line AB sends O to C'. Let D, E, F be the midpoints of sides BC, CA, AB respectively. Then $H(G, -1/2)$ sends ΔABC to ΔDEF and $H(O, 2)$ sends ΔDEF to $\Delta A'B'C'$. Since $(-1/2)\times 2 \neq 1$, by fact 5,

$$H(O, 2) \circ H(G, -1/2) = H(X, -1)$$

for some point X. Since the composition on the left side sends ΔABC to $\Delta A'B'C'$, segments AA', BB', CC' concur at X and in fact X is their common midpoint.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 31, 2008.***

Problem 306. Prove that for every integer $n \geq 48$, every cube can be decomposed into n smaller cubes, where every pair of these small cubes does not have any common interior point and has possibly different sidelengths.

Problem 307. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a polynomial with real coefficients such that $a_0 \neq 0$ and for all real x,

$$f(x)\,f(2x^2) = f(2x^3+x).$$

Prove that $f(x)$ has no real root.

Problem 308. Determine (with proof) the greatest positive integer $n > 1$ such that the system of equations

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+n)^2 + y_n^2.$$

has an integral solution $(x, y_1, y_2, \cdots, y_n)$.

Problem 309. In acute triangle ABC, $AB > AC$. Let H be the foot of the perpendicular from A to BC and M be the midpoint of AH. Let D be the point where the incircle of ΔABC is tangent to side BC. Let line DM intersect the incircle again at N. Prove that $\angle BND = \angle CND$.

Problem 310. (*Due to Pham Van Thuan*) Prove that if p, q are positive real numbers such that $p + q = 2$, then

$$3p^qq^p + p^pq^q \leq 4.$$

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Solutions

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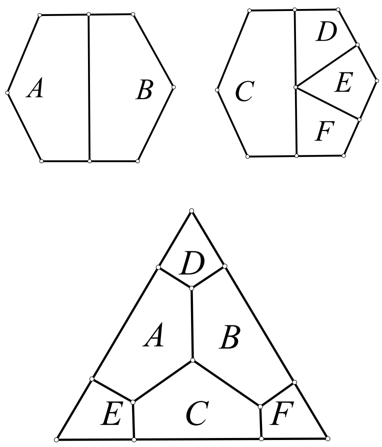
Problem 301. Prove that it is possible to decompose two congruent regular hexagons into a total of six pieces such that they can be rearranged to form an equilateral triangle with no pieces overlapping.

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy).



Commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil), **Glenier L. BELLO- BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **KONG Catherine Wing Yan** (G.T. Ellen Yeung College, Grade 9), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **PUN Ying Anna** (HKU Math Year 3).

Problem 302. Let ℤ denotes the set of all integers. Determine (with proof) all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all x, y in ℤ, we have $f(x+f(y)) = f(x) - y$.
(*Source:2004 Spanish Math Olympiad*)

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **Ozgur KIRCAK** (Jahja Kemal College, Teacher, Skopje, Macedonia), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education), **PUN Ying Anna** (HKU Math Year 3), **Salem MALIKIĆ** (Sarajevo College, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

Assume there is a function f satisfying

$$f(x+f(y)) = f(x) - y. \quad (*)$$

If $f(a) = f(b)$, then

$$f(x)-a = f(x+f(a)) = f(x+f(b)) = f(x)-b,$$

which implies $a = b$, i.e. f is injective. Taking $y = 0$ in (*), $f(x+f(0)) = f(x)$. By injectivity, we see $f(0) = 0$. Taking $x=0$ in (*), we get

$$f(f(y)) = -y. \quad (**)$$

Applying f to both sides of (*) and using (**), we have

$$f(f(x) - y) = f(f(x+f(y)) = -x - f(y).$$

Taking $x = 0$ in this equation, we get

$$f(-y) = -f(y). \quad (***)$$

Using (**), (*) and (***), we get

$$f(x+y) = f(x+f(f(-y)) = f(x) - f(-y) = f(x) + f(y).$$

Thus, f satisfies the <u>*Cauchy equation*</u>. By mathematical induction and (***), $f(n) = n\,f(1)$ for every integer n. Taking $n = f(1)$ in the last equation and $y = 1$ into (**), we get $f(1)^2 = -1$. This yields a contradiction.

Problem 303. In base 10, let N be a positive integer with all digits nonzero. Prove that there do not exist two permutations of the digits of N, forming numbers that are different (integral) powers of two.
(*Source:2004 Spanish Math Olympiad*)

Solution. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Victor FONG** (CUHK Math Year 2), **G.R.A. 20 Problem Solving Group** (Roma, Italy), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education) and **PUN Ying Anna** (HKU Math Year 3).

Assume there exist two permutations of the digits of N, forming the numbers 2^k and 2^m for some positive integers k and m with $k > m$. Then $2^k < 10\times 2^m$. So $k \leq m+3$.

Since every number is congruent to its sum of digits (mod 9), we get $2^k \equiv 2^m$ (mod 9). Since 2^m and 9 are relatively prime, we get $2^{k-m} \equiv 1$ (mod 9). However, $k - m = 1$, 2 or 3, which contradicts $2^{k-m} \equiv 1$ (mod 9).

Problem 304. Let M be a set of 100 distinct lattice points (i.e. coordinates are integers) chosen from the x-y coordinate plane. Prove that there are at most 2025 rectangles whose vertices are in M and whose sides are parallel to the x-axis or the y-axis.
(*Source: 2003 Chinese IMO Team Training Test*)

Solution 1. **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D\`Elhuyar, Spain) and **PUN Ying Anna** (HKU Math Year 3).

Let O be a point in M. We say a rectangle is <u>*good*</u> if all its sides are parallel to the x or y-axis and all its vertices are in M, one of which is O. We claim there are at most 81 good rectangles. (Once the claim is proved, we see there can only be at most $(81\times 100)/4=2025$ desired rectangles.

The division by 4 is due to such rectangle has 4 vertices, hence counted 4 times).

For the proof of the claim, we may assume O is the origin of the plane. Suppose the x-axis contains m points in M other than O and the y-axis contains n points in M other than O. For a point P in M not on either axis, it can only be a vertex of at most one good rectangle. There are at most $99-m-n$ such point P and every good rectangle has such a vertex.

If $m+n \ge 18$, then there are at most $99 - m - n \le 81$ good rectangles. Otherwise, $m+n \le 17$. Now every good rectangle has a vertex on the x-axis and a vertex on the y-axis other than O. So there are at most $mn \le (m+n)^2/4 < 81$ rectangles by the *AM-GM* inequality. The claim follows.

Solution 2. **G.R.A. 20 Problem Solving Group** (Roma, Italy).

Let $f(x) = x(x-1)/2$. We will prove that if there are N lattice points, there are at most $[f(N^{1/2})]^2$ such rectangles. For $N = 100$, we have $[f(10)]^2 = 45^2 = 2025$ (this bound is attained when the 100 points form a 10×10 square).

Suppose the N points are distributed on m lines parallel to an axis. Say the number of points in the m lines are $r_1, r_2, \ldots, r_m$, arranged in increasing order. Now the two lines with r_i and r_j points can form no more than $f(\min\{r_i, r_j\})$ rectangles. Hence, the number of rectangles is at most

$$\sum_{1\le i<j\le m} f(\min\{r_i,r_j\}) = \sum_{i=1}^{m-1}(m-i)f(r_i)$$
$$\le \sum_{i=1}^{m-1}(m-i)f\left(\frac{N}{m}\right) = f(m)f\left(\frac{N}{m}\right)$$
$$\le \left(f(\sqrt{N})\right)^2.$$

The second inequality follows by expansion and usage of the *AM-GM* inequality. The first one can be proved by expanding and simplifying it to

$$2m\sum_{i=1}^{m-1}(m-i)r_i(r_i-1) \le (m-1)\sum_{i=1}^{m} r_i \sum_{i=1}^{m}(r_i-1). \quad (*)$$

We will prove this by induction on m. For $m=2$, $4r_1(r_1-1) \le (r_1+r_2)(r_1-1+r_2-1)$ follows from $1 \le r_1 \le r_2$. For the inductive step, we suppose (*) is true. To do the $(m+1)$-st case of (*), observe that $r_i \le r_{m+1}$ implies

$$m\sum_{i=1}^{m} r_i(r_i-1) \le m(r_{m+1}-1)\sum_{i=1}^{m} r_i,$$

$$m\sum_{i=1}^{m} r_i(r_i-1) \le mr_{m+1}\sum_{i=1}^{m}(r_i-1),$$

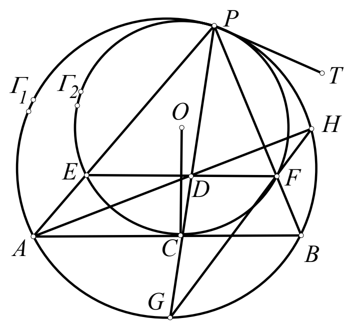
$$2\sum_{i=1}^{m}(m+1-i)r_i(r_i-1)$$
$$\le mr_{m+1}(r_{m+1}-1) + \sum_{i=1}^{m} r_i \sum_{i=1}^{m}(r_i-1).$$

Let $L(m)$ and $R(m)$ denote the left and right sides of (*) respectively. Adding the last three inequalities, it turns out we get $L(m+1) - L(m) \le R(m+1) - R(m)$. Now (*) holds, so $L(m) \le R(m)$. Adding these, we get $L(m+1) \le R(m+1)$.

Commended solvers: **Victor FONG** (CUHK Math Year 2), **O Kin Chit Alex** (G.T. Ellen Yeung College) and **Raúl A. SIMON** (Santiago, Chile).

Problem 305. A circle Γ_2 is internally tangent to the circumcircle Γ_1 of ΔPAB at P and side AB at C. Let E, F be the intersection of Γ_2 with sides PA, PB respectively. Let EF intersect PC at D. Lines PD, AD intersect Γ_1 again at G, H respectively. Prove that F, G, H are collinear.

Solution. **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Glenier L. BELLO-BURGUET** (I.E.S. Hermanos D`Elhuyar, Spain), **NGUYEN Tho Tung** (High School for Gifted Education, Ha Noi University of Education) and **PUN Ying Anna** (HKU Math Year 3).



Let PT be the external tangent to both circles at P. We have

$$\angle PAB = \angle BPT = \angle PEF,$$

which implies $EF \| AB$. Let O be the center of Γ_2. Since $OC \perp AB$ (because AB is tangent to Γ_2 at C), we deduce that $OC \perp EF$ and therefore OC is the perpendicular bisector of EF. Hence C is the midpoint of arc ECF. Then PC bisects $\angle EPF$. On the other hand,

$$\angle HDF = \angle HAB = \angle HPB = \angle HPF,$$

which implies H, P, D, F are concyclic. Therefore,

$$\angle DHF = \angle DPF = \angle EPD$$
$$= \angle APG = \angle AHG = \angle DHG,$$

which implies F, G, H are collinear.

Remarks. A few solvers got $EF \| AB$ by observing there is a homothety with center P sending Γ_2 to Γ_1 so that E goes to A and F goes to B.

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Olympiad Corner

(continued from page 1)

Problem 4. Find all functions f: $(0,\infty) \to (0,\infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 5. Let n and k be positive integers with $k \ge n$ and $k-n$ an even number. Let $2n$ lamps labeled 1, 2, …, $2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamp 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ and $2n$ is ever switched on.

Determine the ratio N/M.

Problem 6. Let $ABCD$ be a convex quadrilateral with $|BA| \ne |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

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Problem 1. Let again O and R be the circumcentre and circumradius. Consider the vectors

$$\overrightarrow{OA}=\mathbf{a},\quad \overrightarrow{OB}=\mathbf{b},\quad \overrightarrow{OC}=\mathbf{c},\quad \text{where}\quad \mathbf{a}^2=\mathbf{b}^2=\mathbf{c}^2=R^2.$$

It is well known that $\overrightarrow{OH}=\mathbf{a}+\mathbf{b}+\mathbf{c}$. Accordingly,

$$\overrightarrow{A_0H}=\overrightarrow{OH}-\overrightarrow{OA_0}=(\mathbf{a}+\mathbf{b}+\mathbf{c})-\frac{\mathbf{b}+\mathbf{c}}{2}=\frac{2\mathbf{a}+\mathbf{b}+\mathbf{c}}{2},$$

and

$$OA_1^2=OA_0^2+A_0A_1^2=OA_0^2+A_0H^2=\left(\frac{\mathbf{b}+\mathbf{c}}{2}\right)^2+\left(\frac{2\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}\right)^2$$

$$=\frac14(\mathbf{b}^2+2\mathbf{bc}+\mathbf{c}^2)+\frac14(4\mathbf{a}^2+4\mathbf{ab}+4\mathbf{ac}+\mathbf{b}^2+2\mathbf{bc}+\mathbf{c}^2)=2R^2+(\mathbf{ab}+\mathbf{ac}+\mathbf{bc});$$

here $\mathbf{ab}$, $\mathbf{bc}$, etc. denote dot products of vectors. We get the same for the distances OA_2, OB_1, OB_2, OC_1 and OC_2.

Problem 2 (a) We start with the substitution

$$\frac{x}{x-1}=a,\quad \frac{y}{y-1}=b,\quad \frac{z}{z-1}=c,\quad \text{i.e.,}\quad x=\frac{a}{a-1},\quad y=\frac{b}{b-1},\quad z=\frac{c}{c-1}.$$

The inequality to be proved reads $a^2+b^2+c^2\ge 1$. The new variables are subject to the constraints $a,b,c\ne 1$ and the following one coming from the condition $xyz=1$,

$$(a-1)(b-1)(c-1)=abc.$$

This is successively equivalent to

$$a+b+c-1=ab+bc+ca,$$
$$2(a+b+c-1)=(a+b+c)^2-(a^2+b^2+c^2),$$
$$a^2+b^2+c^2-2=(a+b+c)^2-2(a+b+c),$$
$$a^2+b^2+c^2-1=(a+b+c-1)^2.$$

Thus indeed $a^2+b^2+c^2\ge 1$, as desired.

(b) From the equation $a^2+b^2+c^2-1=(a+b+c-1)^2$ we see that the proposed inequality becomes an equality if and only if both sums $a^2+b^2+c^2$ and $a+b+c$ have value 1. The first of them is equal to $(a+b+c)^2-2(ab+bc+ca)$. So the instances of equality are described by the system of two equations

$$a+b+c=1,\qquad ab+bc+ca=0$$

plus the constraint $a,b,c\ne 1$. Elimination of c leads to $a^2+ab+b^2=a+b$. Let $m=b/a$. Then $a^2(1+m+m^2)=a(1+m)$, which implies $a=\frac{1+m}{1+m+m^2}$, $b=\frac{m+m^2}{1+m+m^2}$, $c=\frac{-m}{1+m+m^2}$. Now $-\frac{1}{m}<-\frac{m}{1+m+m^2}=c<-\frac{1}{m+2}$ if $m>1$. Taking m to range over all even positive integers, we get infinitely many different c's, hence infinitely many triples of rational numbers a,b,c, then infinitely many corresponding triples of rational numbers x,y,z.

Problem 3 Let $p\equiv 1 \pmod 8$ be a prime. The congruence $x^2\equiv -1 \pmod p$ has two solutions in $[1,p-1]$ whose sum is p. If n is the smaller one of them then p divides n^2+1 and $n\le (p-1)/2$. We show that $p>2n+\sqrt{10n}$.

Let $n=(p-1)/2-\ell$ where $\ell\ge 0$. Then $n^2\equiv -1 \pmod p$ gives

$$\left(\frac{p-1}{2}-\ell\right)^2\equiv -1 \pmod p\qquad\text{or}\qquad (2\ell+1)^2+4\equiv 0 \pmod p.$$

Thus $(2\ell+1)^2+4=rp$ for some $r\ge 0$. As $(2\ell+1)^2\equiv 1\equiv p \pmod 8$, we have $r\equiv 5 \pmod 8$, so that $r\ge 5$. Hence $(2\ell+1)^2+4\ge 5p$, implying $\ell\ge\left(\sqrt{5p-4}-1\right)/2$. Set $\sqrt{5p-4}=u$ for clarity; then $\ell\ge (u-1)/2$. Therefore

$$n=\frac{p-1}{2}-\ell\le\frac12(p-u).$$

Combined with $p=(u^2+4)/5$, this leads to $u^2-5u-10n+4\ge 0$. Solving this quadratic inequality with respect to $u\ge 0$ gives $u\ge\left(5+\sqrt{40n+9}\right)/2$. So the estimate $n\le (p-u)/2$ leads to

$$p\ge 2n+u\ge 2n+\frac12\left(5+\sqrt{40n+9}\right)>2n+\sqrt{10n}.$$

Since there are infinitely many primes of the form $8k+1$, it follows easily that there are also infinitely many n with the stated property.

Problem 4 Let f satisfy the given condition. Setting $p=q=r=s=1$ yields $f(1)^2=f(1)$ and hence $f(1)=1$. Now take any $x>0$ and set $p=x$, $q=1$, $r=s=\sqrt{x}$ to obtain

$$\frac{f(x)^2+1}{2f(x)}=\frac{x^2+1}{2x}.$$

This recasts into

$$xf(x)^2+x=x^2f(x)+f(x),$$
$$\big(xf(x)-1\big)\big(f(x)-x\big)=0.$$

And thus,

$$\text{for every } x>0,\quad \text{either } f(x)=x \text{ or } f(x)=\frac1x. \qquad (1)$$

Obviously, if

$$f(x)=x \text{ for all } x>0\qquad\text{or}\qquad f(x)=\frac1x \text{ for all } x>0 \qquad (2)$$

then the condition of the problem is satisfied. We show that actually these two functions are the only solutions.

So let us assume that there exists a function f satisfying the requirement, other than those in (2). Then $f(a)\ne a$ and $f(b)\ne 1/b$ for some $a,b>0$. By (1), these values must be $f(a)=1/a$, $f(b)=b$. Applying now the equation with $p=a$, $q=b$, $r=s=\sqrt{ab}$ we obtain $(a^{-2}+b^2)/2f(ab)=(a^2+b^2)/2ab$; equivalently,

$$f(ab)=\frac{ab(a^{-2}+b^2)}{a^2+b^2}. \qquad (3)$$

We know however (see (1)) that $f(ab)$ must be either ab or $1/ab$. If $f(ab)=ab$ then by (3) $a^{-2}+b^2=a^2+b^2$, so that $a=1$. But, as $f(1)=1$, this contradicts the relation $f(a)\ne a$. Likewise, if $f(ab)=1/ab$ then (3) gives $a^2b^2(a^{-2}+b^2)=a^2+b^2$, whence $b=1$, in contradiction to $f(b)\ne 1/b$. Thus indeed the functions listed in (2) are the only two solutions.

Problem 5 A sequence of k switches ending in the state as described in the problem statement (lamps $1, \dots, n$ *on*, lamps $n+1, \dots, 2n$ *off*) will be called an *admissible process*. If, moreover, the process does not touch the lamps $n+1, \dots, 2n$, it will be called *restricted*. So there are N admissible processes, among which M are restricted.

In every admissible process, restricted or not, each one of the lamps $1, \dots, n$ goes from *off* to *on*, so it is switched an odd number of times; and each one of the lamps $n+1, \dots, 2n$ goes from *off* to *off*, so it is switched an even number of times.

Notice that $M > 0$; i.e., restricted admissible processes do exist (it suffices to switch each one of the lamps $1, \dots, n$ just once and then choose one of them and switch it $k-n$ times, which by hypothesis is an even number).

Consider any restricted admissible process $\mathbf{p}$. Take any lamp ℓ, $1 \le \ell \le n$, and suppose that it was switched k_ℓ times. As noticed, k_ℓ must be odd. Select arbitrarily an even number of these k_ℓ switches and replace each of them by the switch of lamp $n+\ell$. This can be done in $2^{k_\ell - 1}$ ways (because a k_ℓ-element set has $2^{k_\ell - 1}$ subsets of even cardinality). Notice that $k_1 + \dots + k_n = k$.

These actions are independent, in the sense that the action involving lamp ℓ does not affect the action involving any other lamp. So there are $2^{k_1-1} \cdot 2^{k_2-1} \cdots 2^{k_n-1} = 2^{k-n}$ ways of combining these actions. In any of these combinations, each one of the lamps $n+1, \dots, 2n$ gets switched an even number of times and each one of the lamps $1, \dots, n$ remains switched an odd number of times, so the final state is the same as that resulting from the original process $\mathbf{p}$.

This shows that every restricted admissible process $\mathbf{p}$ can be modified in 2^{k-n} ways, giving rise to 2^{k-n} distinct admissible processes (with all lamps allowed).

Now we show that every admissible process $\mathbf{q}$ can be achieved in that way. Indeed, it is enough to replace every switch of a lamp with a label $\ell > n$ that occurs in $\mathbf{q}$ by the switch of the corresponding lamp $\ell - n$; in the resulting process $\mathbf{p}$ the lamps $n+1, \dots, 2n$ are not involved.

Switches of each lamp with a label $\ell > n$ had occurred in $\mathbf{q}$ an even number of times. So the performed replacements have affected each lamp with a label $\ell \le n$ also an even number of times; hence in the overall effect the final state of each lamp has remained the same. This means that the resulting process $\mathbf{p}$ is admissible—and clearly restricted, as the lamps $n+1, \dots, 2n$ are not involved in it any more.

If we now take process $\mathbf{p}$ and reverse all these replacements, then we obtain process $\mathbf{q}$. These reversed replacements are nothing else than the modifications described in the foregoing paragraphs.

Thus there is a one to (2^{k-n}) correspondence between the M restricted admissible processes and the total of N admissible processes. Therefore $N/M = 2^{k-n}$.

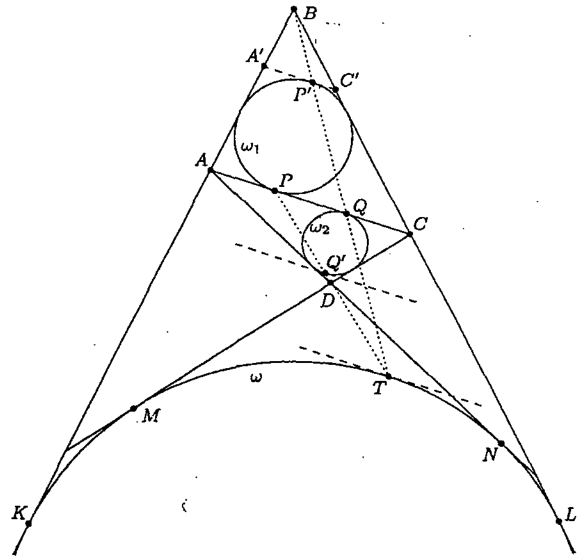
Problem 6. The proof below is based on two known facts.

Lemma 1. Given a convex quadrilateral $ABCD$, suppose that there exists a circle which is inscribed in angle ABC and tangent to the extensions of line segments AD and CD. Then $AB + AD = CB + CD$.

Proof. The circle in question is tangent to each of the lines AB, BC, CD, DA, and the respective points of tangency K, L, M, N are located as with circle ω in the figure. Then

$$AB + AD = (BK - AK) + (AN - DN), \quad CB + CD = (BL - CL) + (CM - DM).$$

Also $BK = BL$, $DN = DM$, $AK = AN$, $CL = CM$ by equalities of tangents. It follows that $AB + AD = CB + CD$.



For brevity, in the sequel we write "excircle AC" for the excircle of a triangle with side AC which is tangent to line segment AC and the extensions of the other two sides.

Lemma 2. The incircle of triangle ABC is tangent to its side AC at P. Let PP' be the diameter of the incircle through P, and let line BP' intersect AC at Q. Then Q is the point of tangency of side AC and excircle AC.

Proof. Let the tangent at P' to the incircle ω_1 meet BA and BC at A' and C'. Now ω_1 is the excircle $A'C'$ of triangle $A'BC'$, and it touches side $A'C'$ at P'. Since $A'C' \parallel AC$, the homothety with centre B and ratio BQ/BP' takes ω_1 to the excircle AC of triangle ABC. Because this homothety takes P' to Q, the lemma follows.

Recall also that if the incircle of a triangle touches its side AC at P, then the tangency point Q of the same side and excircle AC is the unique point on line segment AC such that $AP = CQ$.

We pass on to the main proof. Let ω_1 and ω_2 touch AC at P and Q, respectively; then $AP = (AC + AB - BC)/2$, $CQ = (CA + CD - AD)/2$. Since $AB - BC = CD - AD$ by Lemma 1, we obtain $AP = CQ$. It follows that in triangle ABC side AC and excircle AC are tangent at Q. Likewise, in triangle ADC side AC and excircle AC are tangent at P. Note that $P \neq Q$ as $AB \neq BC$.

Let PP' and QQ' be the diameters perpendicular to AC of ω_1 and ω_2, respectively. Then Lemma 2 shows that points B, P' and Q are collinear, and so are points D, Q' and P.

Consider the diameter of ω perpendicular to AC and denote by T its endpoint that is closer to AC. The homothety with centre B and ratio BT/BP' takes ω_1 to ω. Hence B, P' and T are collinear. Similarly, D, Q' and T are collinear since the homothety with centre D and ratio $-DT/DQ'$ takes ω_2 to ω.

We infer that points T, P' and Q are collinear, as well as T, Q' and P. Since $PP' \parallel QQ'$, line segments PP' and QQ' are then homothetic with centre T. The same holds true for circles ω_1 and ω_2 because they have PP' and QQ' as diameters. Moreover, it is immediate that T lies on the same side of line PP' as Q and Q', hence the ratio of homothety is positive. In particular ω_1 and ω_2 are not congruent.

In summary, T is the centre of a homothety with positive ratio that takes circle ω_1 to circle ω_2. This completes the solution, since the only point with the mentioned property is the intersection of the the common external tangents of ω_1 and ω_2.