Mathematical Excalibur

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Olympiad Corner

Below were the problems of the 2007Estonian IMO Team Selection Contest.

First Day

Problem 1. On the control board of a nuclear station, there are n electric switches ($n > 0$), all in one row. Each switch has two possible positions: up and down. The switches are connected to each other in such a way that, whenever a switch moves down from its upper position, its right neighbor (if it exists) automatically changes position. At the beginning, all switches are down. The operator of the board first changes the position of the leftmost switch once, then the position of the second leftmost switch twice etc., until eventually he changes the position of the rightmost switch n times. How many switches are up after all these operations?

Problem 2. Let D be the foot of the altitude of triangle ABC drawn from vertex A. Let E and F be points symmetric to D with respect to lines AB and AC, respectively. Let R_1 and R_2 be

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 25, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Square It!

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Inequalities involving square roots of the form

$$\sqrt{A}+\sqrt{B}+\sqrt{C}+\sqrt{D}\le k$$

can be solved using the Cauchy-Schwarz inequality. However, solving inequalities of the following form

$$\sqrt{A}+\sqrt{B}+\sqrt{C}+\sqrt{D}\ge k$$

is far from straightforward. In this article, we will look at such problems. We will solve them by squaring and making more delicate use of the Cauchy-Schwarz inequality.

Example 1. Three nonnegative real numbers x, y and z satisfy $x^2+y^2+z^2=1$. Prove that

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\sqrt{1-\left(\frac{y+z}{2}\right)^2}+\sqrt{1-\left(\frac{z+x}{2}\right)^2}\ge\sqrt{6}.$$

Solution. Squaring both sides of the inequality and simplifying, we get the equivalent inequality

$$\sum_{cyclic}\sqrt{1-\left(\frac{x+y}{2}\right)^2}\sqrt{1-\left(\frac{y+z}{2}\right)^2}\ge\frac{7}{4}+\frac{xy+yz+zx}{4},$$

where

$$\sum_{cyclic}f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y).$$

Notice that

$$1-\left(\frac{x+y}{2}\right)^2=\frac{x^2+y^2+(z^2+1)}{2}-\left(\frac{x+y}{2}\right)^2$$
$$=\frac{(x-y)^2}{4}+\frac{z^2+1}{2}.$$

By the Cauchy-Schwarz inequality,

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}\sqrt{1-\left(\frac{y+z}{2}\right)^2}$$
$$\ge\frac{(x-y)(z-y)}{4}+\frac{\sqrt{(z^2+1)(x^2+1)}}{2}$$
$$\ge\frac{y^2+xz-yz-xy}{4}+\frac{zx+1}{2}.$$

Similarly, we obtain two other such inequalities. Multiplying each of them by 2, adding them together, simplifying and finally using $x^2+y^2+z^2=1$, we get the equivalent inequality in the beginning of this solution.

Example 2. For a, b, $c>0$, prove that

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}$$
$$\ge 2\sqrt{1+\frac{abc}{(a+b)(b+c)(c+a)}}.$$

Solution. Multiplying both sides by $\sqrt{(a+b)(b+c)(c+a)}$, we have to show

$$\sum_{cyclic}\sqrt{a(c+a)(a+b)}$$
$$\ge 2\sqrt{(a+b+c)(ab+bc+ca)}.$$

Squaring both sides, we get the equivalent inequality

$$\sum_{cyclic}a^3+2\sum_{cyclic}(a+b)\sqrt{ab(a+c)(b+c)}$$
$$\ge 3\sum_{cyclic}ab(a+b)+9abc. \qquad (*)$$

By the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$(a+b)\sqrt{ab(a+c)(b+c)}$$
$$\ge(a+b)\sqrt{ab(\sqrt{ab}+c)^2}$$
$$=(a+b)(\sqrt{ab}+c)\sqrt{ab}$$
$$=ab(a+b)+(a+b)c\sqrt{ab}$$
$$\ge ab(a+b)+2abc.$$

Using this, we have

$$\sum_{cyclic}a^3+2\sum_{cyclic}(a+b)\sqrt{ab(c+a)(c+b)}$$
$$\ge\sum_{cyclic}a^3+2\sum_{cyclic}ab(a+b)+12abc.$$

Comparing with (*), we need to show

$$\sum_{cyclic}a^3-\sum_{cyclic}ab(a+b)+3abc\ge 0.$$

This is just Schur's inequality

$$\sum_{cyclic}a(a-b)(a-c)\ge 0.$$

(See *Math. Excalibur*, vol.10, no.5, p.2)

From the last example, we saw that other than the Cauchy-Schwarz inequality, we might need to recall Schur's inequality

$$\sum_{cyclic} x^r(x-y)(x-z) \ge 0.$$

Here we will also point out a common variant of Schur's inequality, namely

$$\sum_{cyclic} x^r(y+z)(x-y)(x-z) \ge 0.$$

This variant can be proved in the same way as Schur's inequality (again see Math. Excalibur, vol.10, no.5, p.2). Both inequalities become equality if and only if either the variables are all equal or one of them is zero, while the other two are equal. In the next two examples, we will use these.

Example 3. Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Prove that

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \ge \sqrt{3}.$$

When does equality occur?

Solution. Squaring both sides of the inequality and using

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 1-2(ab+bc+ca),$$

we get the equivalent inequality

$$\sum_{cyclic}\sqrt{a+(b-c)^2}\sqrt{b+(c-a)^2} \ge 3(ab+bc+ca).$$

By the Cauchy-Schwarz inequality,

$$\sqrt{a+(b-c)^2}\sqrt{b+(c-a)^2}$$
$$= \sqrt{(b-c)^2+(a+b+c)a}\sqrt{(c-a)^2+(a+b+c)b}$$
$$\ge |(b-c)(c-a)| + (a+b+c)\sqrt{ab}.$$

Similarly, we can obtain two other such inequalities. Adding them together, the right side is

$$\sum_{cyclic}|(b-c)(c-a)| + (a+b+c)\sum_{cyclic}\sqrt{ab}.$$

By the triangle inequality and the case $r = 0$ of Schur's inequality, we get

$$\sum_{cyclic}|(b-c)(c-a)| \ge \left|\sum_{cyclic}(b-c)(c-a)\right| = \sum_{cyclic}(c-b)(c-a) = (a^2+b^2+c^2)-(ab+bc+ca). \qquad (**)$$

Thus, to finish, it will be enough to show

$$a^2+b^2+c^2+(a+b+c)(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \ge 4(ab+bc+ca).$$

Now we make the substitutions

$$x=\sqrt{a},\quad y=\sqrt{b} \quad\text{and}\quad z=\sqrt{c}.$$

In terms of x, y, z, the last inequality becomes

$$\sum_{cyclic}(x^4+x^3y+x^3z+x^2yz-4x^2y^2) \ge 0. \quad (***)$$

Since the terms are of degree 4, we consider the case $r = 2$ of Schur's inequality, which is

$$\sum_{cyclic}x^2(x-y)(x-z)$$
$$= \sum_{cyclic}(x^4-x^3y-x^3z+x^2yz) \ge 0.$$

This is not quite equal to (***). So next (due to degree 4 consideration again), we will look at the case $r = 1$ of the variant

$$\sum_{cyclic}x(y+z)(x-y)(x-z)$$
$$= \sum_{cyclic}(x^3y+x^3z-2x^2y^2) \ge 0.$$

Readily we see (***) is just the sum of Schur's inequality with twice its variant.

Finally, tracing back, we see equality occurs if and only if $a = b = c = 1/3$ or one of them is 0, while the other two are equal to 1/2.

Example 4. Three nonnegative real numbers a, b, c satisfy $a + b + c = 2$. Prove that

$$\sqrt{\frac{a+b}{2}-ab}+\sqrt{\frac{b+c}{2}-bc}+\sqrt{\frac{c+a}{2}-ca} \ge \sqrt{2}.$$

Solution. Squaring both sides of the inequality and using $a + b + c = 2$, we get the equivalent inequality

$$\sum_{cyclic}\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)} \ge \frac{ab+bc+ca}{2}.$$

Note that

$$\frac{a+b}{2}-ab = \frac{2(a+b)-(a+b)^2+(a-b)^2}{4}$$
$$= \frac{(a-b)^2+(2-a-b)(a+b)}{4}$$
$$= \frac{(a-b)^2}{4}+\frac{c(a+b)}{4}.$$

Applying twice the Cauchy-Schwarz inequality, we have

$$\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)}$$
$$\ge \frac{|(a-b)(b-c)|}{4}+\frac{\sqrt{ca(a+b)(b+c)}}{4}$$
$$\ge \frac{1}{4}\left(|(a-b)(b-c)|+\sqrt{ca(b+\sqrt{ca})^2}\right)$$
$$= \frac{1}{4}\left(|(a-b)(b-c)|+\sqrt{abc}\sqrt{b}+ca\right).$$

Similarly, we can obtain two other such inequalities. Adding them together and using (**) in example 3, we get

$$4\sum_{cyclic}\sqrt{\left(\frac{a+b}{2}-ab\right)\left(\frac{b+c}{2}-bc\right)}$$
$$\ge \sum_{cyclic}\left(|(a-b)(b-c)|\right)+\sqrt{abc}\sum_{cyclic}\sqrt{b}+\sum_{cyclic}ca$$
$$\ge a^2+b^2+c^2+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Substituting

$$x=\sqrt{a},\quad y=\sqrt{b} \quad\text{and}\quad z=\sqrt{c}$$

and using Schur's inequality and its variant, we have

$$a^2+b^2+c^2+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})$$
$$= x^4+y^4+z^4+x^2yz+xy^2z+xyz^2$$
$$\ge \sum_{cyclic}\left(x^3y+x^3z\right)$$
$$\ge 2\sum_{cyclic}x^2y^2 = 2(ab+bc+ca).$$

Combining this with the last displayed inequalities, we can obtain the equivalent inequality in the beginning of this solution.

To conclude this article, we will give two exercises for the readers to practice.

Exercise 1. Three nonnegative real numbers x, y and z satisfy $x^2+y^2+z^2=1$. Prove that

$$\sum_{cyclic}\sqrt{1-xy}\sqrt{1-yz} \ge 2.$$

Exercise 2. Three nonnegative real numbers x, y and z satisfy $x + y + z = 1$. Prove that

$$x\sqrt{1-yz}+y\sqrt{1-zx}+z\sqrt{1-xy} \ge \frac{2\sqrt{2}}{3}.$$

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***February 25, 2008.***

Problem 291. Prove that if a convex polygon lies in the interior of another convex polygon, then the perimeter of the inner polygon is less than the perimeter of the outer polygon.

Problem 292. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers with no two of them are consecutive. For every m = 1, 2, 3, …, let $S_m = k_1+k_2+\cdots+k_m$. Prove that for every positive integer n, the interval $[S_n, S_{n+1})$ contains at least one perfect square number.
(*Source: 1996 Shanghai Math Contest*)

Problem 293. Let CH be the altitude of triangle ABC with $\angle ACB = 90°$. The bisector of $\angle BAC$ intersects CH, CB at P, M respectively. The bisector of $\angle ABC$ intersects CH, CA at Q, N respectively. Prove that the line passing through the midpoints of PM and QN is parallel to line AB.

Problem 294. For three nonnegative real numbers x, y, z satisfying the condition $xy + yz + zx = 3$, prove that

$$x^2 + y^2 + z^2 + 3xyz \ge 6.$$

Problem 295. There are $2n$ distinct points in space, where $n \ge 2$. No four of them are on the same plane. If $n^2 + 1$ pairs of them are connected by line segments, then prove that there are at least n distinct triangles formed.
(*Source: 1989 Chinese IMO team training problem*)

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Solutions

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Problem 286. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that there exists a real number y such that the sum of $\{x_1-y\}, \{x_2-y\}, \ldots, \{x_n-y\}$ is at most $(n-1)/2$.

(Here $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x.)

Can y always be chosen to be one of the x_i's ?

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **HO Kin Fai** (HKUST, Math Year 3), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

For i = 1, 2, …, n, let

$$S_i = \sum_{j=1}^{n} \{x_j - x_i\}.$$

For all real x, $\{x\} + \{-x\} \le 1$ (since the left side equals 0 if x is an integer and equals 1 otherwise). Using this, we have

$$\sum_{i=1}^{n} S_i = \sum_{1\le i<j\le n} (\{x_j - x_i\} + \{x_i - x_j\})$$

$$\le \sum_{1\le i<j\le n} 1 = \frac{n(n-1)}{2}.$$

So the average value of S_i is at most $(n-1)/2$. Therefore, there exists some $y = x_i$ such that S_i is at most $(n-1)/2$.

Problem 287. Determine (with proof) all nonempty subsets A, B, C of the set of all positive integers $\mathbb{Z}^+$ satisfying

(1) $A\cap B = B\cap C = C\cap A = \varnothing$;
(2) $A\cup B\cup C = \mathbb{Z}^+$;
(3) for every $a\in A$, $b\in B$ and $c\in C$, we have $c + a \in A$, $b + c \in B$ and $a + b \in C$.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **HO Kin Fai** (HKUST, Math Year 3), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina) and **Fai YUNG**.

Let the minimal element of C be x. Then $\{1, 2, \ldots, x-1\} \subseteq A\cup B$. Since for every $a\in A$, $b\in B$, we have $x + a\in A$, $b + x\in B$. So all numbers not divisible by x are in $A\cup B$. Then every $c\in C$ is a multiple of x. By (3), the sum of every $a\in A$ and $b\in B$ is a multiple of x.

Assume x = 1. Then $a\in A$, $b\in B$ imply $a+1\in A$, $b + 1\in B$, which lead to $a + b \in A\cap B$ contradicting (1).

Assume x = 2. We may suppose $1\in A$. Then by (3), all odd positive integers are in A. For $b \in B$, we get $1 + b \in C$. Then b is odd, which lead to $b\in A\cap B$ contradicting (1).

Assume $x \ge 4$. Then $\{1,2,3\}\subseteq A\cup B$, say y, $z\in \{1,2,3\}\cap A$. Taking a $b \in B$, we get $y+b$, $z+b \in C$ by (3). Then $(y+b) - (z+b) = y - z$ is a multiple of x. But $|y - z| < x$ leads to a contradiction.

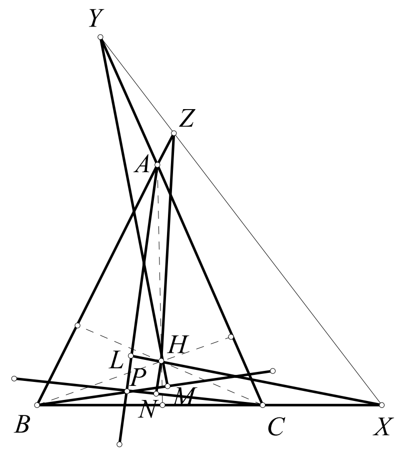
Therefore, x = 3. We claim 1 and 2 cannot both be in A (or both in B). If 1, $2\in A$, then (3) implies $3k + 1$, $3k + 2\in A$ for all $k\in \mathbb{Z}^+$. Taking a $b\in B$, we get $1 + b\in C$, which implies $b = 3k + 2\in A$. Then $b\in A\cap B$ contradicts (1).

Therefore, either $1 \in A$ and $2 \in B$ (which lead to $A = \{1,4,7,\ldots\}$, $B = \{2,5,8,\ldots\}$, $C = \{3,6,9,\ldots\}$) or $2\in A$ and $1\in B$ (which similarly lead to $A = \{2,5,8,\ldots\}$, $B = \{1,4,7,\ldots\}$, $C = \{3,6,9,\ldots\}$).

Problem 288. Let H be the orthocenter of triangle ABC. Let P be a point in the plane of the triangle such that P is different from A, B, C.

Let L, M, N be the feet of the perpendiculars from H to lines PA, PB, PC respectively. Let X, Y, Z be the intersection points of lines LH, MH, NH with lines BC, CA, AB respectively.

Prove that X, Y, Z are on a line perpendicular to line PH.



Solution 1. **Jeff CHEN** (Virginia, USA) and **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England).

Since $XH = LH \perp PA$, $AH \perp CB = XB$, $BH\perp AC=AY$ and $YH=MH \perp BP$, we have respectively (see Math. Excalibur, vol.12, no.3, p.2)

$$XP^2-XA^2=HP^2-HA^2 \qquad (1)$$
$$AX^2-AB^2=HX^2-HB^2 \qquad (2)$$
$$BA^2-BY^2=HA^2-HY^2 \qquad (3)$$
$$YB^2-YP^2=HB^2-HP^2 \qquad (4)$$

Doing (1)+(2)+(3)+(4), we get

$$XP^2-YP^2=XH^2-YH^2,$$

which implies $XY \perp PH$. Similarly, $ZY \perp PH$. So, X, Y, Z are on a line perpendicular to line PH.

Solution 2. **Anna Ying PUN** (HKU, Math Year 2) and **Stephen KIM** (Toronto, Canada)**.**

Set the origin of the coordinate plane at H. For a point J, let (x_J, y_J) denote its coordinates. Since the slope of line PA is $(y_P - y_A)/(x_P - x_A)$, the equation of line HL is

$$(x_P - x_A)x + (y_P - y_A)y = 0. \qquad (1)$$

Since the slope of line HA is y_A/x_A, the equation of line BC is

$$x_A x + y_A y = x_A x_B + y_A y_B. \qquad (2)$$

Let $t = x_A x_B + y_A y_B$. Since point C is on line BC, we get $x_A x_C + y_A y_C = x_A x_B + y_A y_B = t$. Similarly, $x_B x_C + y_B y_C = t$.

Since X is the intersection of lines BC and HL, so the coordinates of X satisfy the sum of equations (1) and (2), that is

$$x_P x + y_P y = t.$$

(Since the slope of line PH is y_P/x_P, this is the equation of a line that is perpendicular to line PH.) Similarly, the coordinates of Y and Z satisfy $x_P x + y_P y = t$. Therefore, X, Y, Z lie on a line perpendicular to line PH.

Commended solvers: **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Problem 289. Let a and b be positive numbers such that $a + b < 1$. Prove that

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \ge \min\left\{\frac{a}{b}, \frac{b}{a}\right\}.$$

Solution. **Samuel Liló ABDALLA** (ITA, São Paulo, Brazil), **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **Anna Ying PUN** (HKU, Math Year 2), **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina), **Simon YAU Chi Keung** (City University of Hong Kong) and **Fai YUNG**.

Since $0 < a, b < a + b < 1$, we have

$$(b-1)^2 + a(2b-a) = b^2+2(a-1)b-a^2+1$$
$$= (b+a-1)^2+2a(1-a) > 0.$$

In case $a \ge b > 0$, we have

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \ge \min\left\{\frac{a}{b}, \frac{b}{a}\right\} = \frac{b}{a}$$

$$\Leftrightarrow \ a(a-1)^2+ab(2a-b)$$
$$\ge b(b-1)^2+ab(2b-a)$$
$$\Leftrightarrow \ (a-b)[(a+b-1)^2+2ab] \ge 0,$$

which is true. In case $b > a > 0$, we have

$$\frac{(a-1)^2 + b(2a-b)}{(b-1)^2 + a(2b-a)} \ge \min\left\{\frac{a}{b}, \frac{b}{a}\right\} = \frac{a}{b}$$

$$\Leftrightarrow \ b(a-1)^2+b^2(2a-b)$$
$$\ge a(b-1)^2+a^2(2b-a)$$
$$\Leftrightarrow \ (b-a)(1-a^2-b^2) \ge 0,$$

which is also true as $a^2 + b^2 < a+b < 1$.

Problem 290. Prove that for every integer a greater than 2, there exist infinitely many positive integers n such that $a^n - 1$ is divisible by n.

Solution 1. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Magdalene College, University of Cambridge, England), **GRA20 Problem Solving Group** (Roma, Italy) and **HO Kin Fai** (HKUST, Math Year 3).

We will show by math induction that $n = (a-1)^k$ for $k = 1, 2, 3, \ldots$ satisfy the requirement. For $k = 1$, since $a - 1 > 1$ and $a \equiv 1 \pmod{a-1}$, so

$$a^{a-1} - 1 \equiv 1^{a-1} - 1 = 0 \pmod{a-1}.$$

Next, suppose case k is true. Then $a^{(a-1)^k} - 1$ is divisible by $(a-1)^k$. For the case $k + 1$, all we need to show is

$$\frac{a^{(a-1)^{k+1}} - 1}{a^{(a-1)^k} - 1} \equiv 0 \pmod{a-1}.$$

Note $b = a^{(a-1)^k} \equiv 1 \pmod{a-1}$. The left side of the above displayed congruence is

$$\frac{b^{a-1}-1}{b-1} = \sum_{k=0}^{a-2} b^k \equiv \sum_{k=0}^{a-2} 1 = a-1 \equiv 0 \pmod{a-1}.$$

This completes the induction.

Solution 2. **Anna Ying PUN** (HKU, Math Year 2) and **Salem MALIKIĆ** (Sarajevo College, 4th Grade, Sarajevo, Bosnia and Herzegovina).

Note $n = 1$ works. We will show if n works, then $a^n - 1 (> 2^n - 1 \ge n)$ also works. If n works, then $a^n - 1 = nk$ for some positive integer k. Then

$$a^{a^n - 1} - 1 = a^{nk} - 1 = (a^n - 1)\sum_{j=0}^{k-1} a^{nj},$$

which shows $a^n - 1$ works.

Comments: Cheung Wang Chi pointed out that interestingly $n = 1$ is the only positive integer such that $2^n - 1$ is divisible by n (denote this by $n \mid 2^n - 1$). [This fact appeared in the 1972 Putnam Exam.-*Ed*.] To see this, he considered a minimal $n > 1$ such that $n \mid 2^n - 1$. He showed if $a, b, q \in \mathbb{Z}^+$ and $a = bq + r$ with $0 \le r < b$, then $2^a - 1 = ((2^b)^q - 1)2^r + (2^r - 1) = (2^b - 1)N + (2^r - 1)$ for some $N \in \mathbb{Z}^+$. Hence,

$$\gcd(2^a-1, 2^b-1) = \gcd(2^b-1, 2^r-1)$$
$$= \cdots = 2^{\gcd(a,b)}-1$$

by the Euclidean algorithm. Since $n | 2^n - 1$ and $n | 2^{\varphi(n)} - 1$ by Euler's theorem, so $n | 2^d - 1$, where $d = \gcd(n, \varphi(n)) \le \varphi(n) < n$. Then $n \mid 2^d - 1$ implies $d > 1$ and $d | 2^d - 1$, contradicting minimality of n.

Commended solvers: **Samuel Liló ABDALLA** (ITA, São Paulo, Brazil) and **Fai YUNG.**

Olympiad Corner

(continued from page 1)

Problem 2. (*Cont.*) the circumradii of triangles BDE and CDF, respectively, and r_1 and r_2 be the inradii of the same triangles. Prove that

$$|S_{ABD} - S_{ACD}| \ge |R_1 r_1 - R_2 r_2|,$$

where S_K is the area of figure K.

Problem 3. Let n be a natural number, $n \ge 2$. Prove that if $(b^n - 1)/(b-1)$ is a prime power for some positive integer b, then n is prime.

Second Day

Problem 4. In square $ABCD$, points E and F are chosen in the interior of sides BC and CD, respectively. The line drawn from F perpendicular to AE passes through the intersection point G of AE and diagonal BD. A point K is chosen on FG such that $AK = EF$. Find $\angle EKF$.

Problem 5. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all reals x and y, $f(x + f(y)) = y + f(x+1)$.

Problem 6. Consider a 10×10 grid. On every move, we color 4 unit squares that lie in the intersection of some two rows and two columns. A move is allowed if at least one of the 4 squares is previously uncolored. What is the *largest* possible number of moves that can be taken to color the whole grid?

(1)

Answer: 1.

Solution 1. Enumerate the switches with numbers 1 to n from left to right. We prove first that the result of two consecutive changes does not depend on the order of the changes. Let x and y be the numbers of the switches changed, $x < y$.

- If there exists a number z such that $x \le z < y$ and switch number z is down then changing the position of x can influence only switches from x to z, changing the position of switch y can influence only this switch and switches right from y. Thus the results of the changes are independent of each other.
- If no such z exists then changing switch number x causes a change of switch number y. After that, switches x to $y-1$ are all down while all switches in the right from them are in the same position as if switch number y were changed. Thus after moving both x and y in either order, switches from x to $y-1$ are down and the switches with larger number are in the position as when switch y were moved twice.

We prove now that, after all operations, precisely the leftmost switch is up. This claim holds trivially for $n = 1$. Assume the claim holding for n switches and consider a board with $n+1$ switches. According to what was proven above, the moves can be performed in arbitrary order. Therefore, first change switch number 2 once, then switch number 3 twice etc., until the last switch n times. By the induction hypothesis, switch number 2 is up and all the others are down. Each switch has to be moved once more; if we do it from right to left then switches $n+1$ to 3 go up, then moving switch 2 down brings all them down and finally switch 1 is moved up. Thus 1 is the only switch remaining up.

Solution 2. Let a_i be the number of times the ith switch changes its position during the whole process. According to the conditions of the problem, each switch moves either when it is moved directly by the operator or its left neighbour moves down. As all switches are down at the beginning, the ith switch moves down $\left\lfloor \frac{a_i}{2} \right\rfloor$ times. Thus $a_1 = 1$ and $a_i = i + \left\lfloor \frac{a_{i-1}}{2} \right\rfloor$ for all $i \ge 2$.

We prove by induction that $a_i = 2(i-1)$ for all $i \ge 2$. As $a_2 = 2 + \left\lfloor \frac{a_1}{2} \right\rfloor = 2 + \left\lfloor \frac{1}{2} \right\rfloor = 2$, this claim holds for $i = 2$. Assuming that it holds for i, we obtain

$$a_{i+1} = i + 1 + \left\lfloor \frac{a_i}{2} \right\rfloor = i + 1 + \left\lfloor \frac{2(i-1)}{2} \right\rfloor = 2i,$$

i.e., the claim holds also for $i+1$.

Altogehter, this shows that a_1 is odd and a_i is even for all $i \ge 2$. Hence, after the process, the first switch is up and all the others are down.

Solution 3. Interpret the position of switches on the board as binary numbers so that the ith switch from the left corresponds to the ith lowest binary digit: being down encodes 0 and being up encodes 1. Changing the ith switch then works like addition of 2^{i-1} modulo 2^n. The initial position encodes number 0 and the final position encodes $1 \cdot 2^0 + 2 \cdot 2^1 + \ldots + n \cdot 2^{n-1}$ modulo 2^n.

We prove by induction that $1 \cdot 2^0 + 2 \cdot 2^1 + \ldots + n \cdot 2^{n-1} \equiv 1 \pmod{2^n}$. If $n = 1$ then this holds. Assume that the claim holds for $n = k$. Multiplying this congruence by 2 gives

$$1 \cdot 2^1 + 2 \cdot 2^2 + \ldots + k \cdot 2^k \equiv 2 \pmod{2^{k+1}}.$$

Adding $2^0 + 2^1 + \ldots + 2^k$ to both sides gives

$$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots + (k+1) \cdot 2^k \equiv 2 + 2^{k+1} - 1 \equiv 1 \pmod{2^{k+1}},$$

i.e., the claim holds for $n = k+1$.

Remark. In Solution 3, one could prove by induction a stronger claim: $1 \cdot 2^0 + 2 \cdot 2^1 + \ldots + n \cdot 2^{n-1} = (n-1) \cdot 2^n + 1$.

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(2)

Solution 1. Consider first the case where D lies between points B and C (see Fig. 14). As $S_{ABD} = \frac{1}{2} \cdot |AD| \cdot |BD|$ and $S_{ACD} = \frac{1}{2} \cdot |AD| \cdot |CD|$, we have

$$S_{ABD} - S_{ACD} = \frac{1}{2} \cdot |AD| \cdot (|BD| - |CD|).$$

Let G be the incentre of triangle BDE and let G' be the projection of G to line BD. Then $|GG'| = r_1$. By symmetry, $\angle BEA = \angle BDA = 90°$, hence quadrangle $BEAD$ is cyclic and line segment AB is its circumdiameter. Thus $|AB| = 2R_1$. As triangles ADB and $GG'B$ are similar, we have $\frac{|AB|}{|AD|} = \frac{|GB|}{|GG'|}$, implying $2R_1 r_1 = |AD| \cdot |GB|$. Let H be the incentre of triangle CDF; then analogously $2R_2 r_2 = |AD| \cdot |HC|$. Hence

$$R_1 r_1 - R_2 r_2 = \frac{1}{2} \cdot |AD| \cdot (|GB| - |HC|).$$

Triangle ADG is isosceles because $\angle ADG = 90° - \frac{1}{2}\angle BDE = 90° - \frac{1}{2}\angle DAG$. Thus $|AD| = |AG|$. Analogously, $|AD| = |AH|$. Thus $|AG| = |AH|$.
Subtracting equality $|AD|^2 + |CD|^2 = |AC|^2$ from $|AD|^2 + |BD|^2 = |AB|^2$ gives $|BD|^2 - |CD|^2 = |AB|^2 - |AC|^2$ which is equivalent to $(|BD| - |CD|) \cdot (|BD| + |CD|) = (|AB| - |AC|) \cdot (|AB| + |AC|)$. Consequently,

$$\big||BD| - |CD|\big| \cdot |BC| = \big||GB| - |HC|\big| \cdot (|AB| + |AC|).$$

As $|BC| < |AB| + |AC|$, we must have $\big||BD| - |CD|\big| \ge \big||GB| - |HC|\big|$, which gives the desired inequality.

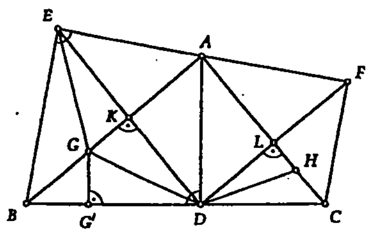

Figure 14

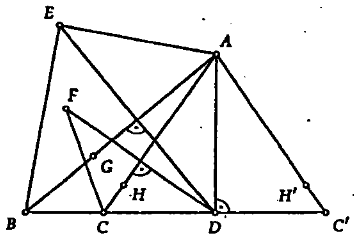

Figure 15

If D does not lie between B and C (see Fig. 15) then assume w.l.o.g. that it is on ray BC. Reflect line segment AC w.r.t. line AD; points C and H transform to some points C' and H', respectively. Now apply the solution above for triangle ABC'. The desired claim follows then by using $|C'D| = |CD|$ and $|H'C'| = |HC|$.

Solution 2. Denote $\angle BAD = \beta$ and $\angle CAD = \gamma$. Then

$$S_{ABD} = \frac{1}{2} \cdot |AD| \cdot |BD| = \frac{1}{2}|AD|^2 \tan\beta.$$

As in Solution 1, show that quadrangle $BEAD$ is cyclic. Let K be the point of intersection of its diagonals. As $R_1 = \frac{|AB|}{2}$, we get $R_1 = \frac{|AD|}{2\cos\beta}$. Furthermore, $r_1 = |GK|$ and $\angle GDK = \frac{\angle BDE}{2} = \frac{\angle BAD}{2} = \frac{\beta}{2}$. Thus $r_1 = |DK| \tan\frac{\beta}{2} = |AD| \sin\beta \tan\frac{\beta}{2}$. Consequently,

$$R_1 r_1 = \frac{|AD|}{2\cos\beta} \cdot |AD| \sin\beta \tan\frac{\beta}{2} = \frac{1}{2}|AD|^2 \tan\beta \tan\frac{\beta}{2}.$$

Analogously we obtain

$$S_{ACD} = \frac{1}{2}|AD|^2 \tan\gamma, \qquad R_2 r_2 = \frac{1}{2}|AD|^2 \tan\gamma \tan\frac{\gamma}{2}.$$

From these equalities, we can conclude that $S_{ABD} - S_{ACD}$ and $R_1r_1 - R_2r_2$ have the same sign since β and γ belong to the first quarter where tan is increasing. W.l.o.g., assume that both are non-negative (otherwise interchange B and C). Then $\beta \geq \gamma$ and the desired inequality is equivalent to $S_{ABD} - R_1r_1 \geq S_{ACD} - R_2r_2$. Now

$$S_{ABD} - R_1r_1 = \frac{1}{2}|AD|^2 \tan\beta \left(1 - \tan\frac{\beta}{2}\right) =$$

$$= \frac{1}{2}|AD|^2 \frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}\left(1-\tan\frac{\beta}{2}\right) = |AD|^2 \frac{\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}},$$

whence

$$S_{ABD} - R_1r_1 = |AD|^2\left(1 - \frac{1}{1+\tan\frac{\beta}{2}}\right)$$

and, analogously,

$$S_{ACD} - R_2r_2 = |AD|^2\left(1 - \frac{1}{1+\tan\frac{\gamma}{2}}\right).$$

By $\beta \geq \gamma$ and tan being increasing, the inequality $S_{ABD} - R_1r_1 \geq S_{ACD} - R_2r_2$ follows.

Solution. Clearly $b \geq 2$. Assume that $\frac{b^n-1}{b-1} = p^l$ where p is prime, then $n \geq 2$ implies $l \geq 1$. If $n = xy$ where both x and y are greater than 1 then consider the representation

$$\frac{b^{xy}-1}{b-1} = \frac{b^{xy}-1}{b^y-1}\cdot\frac{b^y-1}{b-1} = (1+b^y+\ldots+b^{y(x-1)})\cdot\frac{b^y-1}{b-1}.$$

As the product is a power of p, both factors must be powers of p. As $x > 1$ and $y > 1$, both factors are multiples of p. Then $b^y - 1$ is a multiple of p. Thus all addends in the first factor are congruent to 1 modulo p which implies that the first factor is congruent to x modulo p. Hence x is divisible by p. As x was an arbitrary non-trivial factor of n, this shows that $n = p^m$ for a positive integer m.
Now consider the representation

$$\frac{b^{p^m}-1}{b-1} = \frac{b^{p^m}-1}{b^{p^{m-1}}-1}\cdot\ldots\cdot\frac{b^{p^2}-1}{b^p-1}\cdot\frac{b^p-1}{b-1}.$$

Each factor is both greater than 1 and a power of p. As $\frac{b^p-1}{b-1}$ is a positive integral power of p, the numerator is divisible by p, i.e., $b^p \equiv 1 \pmod p$. By Fermat's little theorem, $b^p \equiv b \pmod p$. Thus $b \equiv 1 \pmod p$ and $b-1$ is divisible by p. But then the numerator $b^p - 1$ must be divisible by p^2, i.e., $b^p \equiv 1 \pmod{p^2}$. If $m \geq 2$ then the representation above contains factor $\frac{b^{p^2}-1}{b^p-1} = 1 + b^p + \ldots + b^{p(p-1)}$. On one hand, this is congruent to p modulo p^2 as all addends are congruent to 1. On the other hand, this factor is a power of p while being greater than p, hence it is a multiple of p^2. This contradiction shows that $m = 1$, qed.
Remark 1. Fermat's little theorem can easily be avoided in the solution. Cutting this out from the solution above, it still shows that if $m \geq 2$ then $b^p - 1$ is not divisible by p^2. Continuing from this, we see that $\frac{b^p-1}{b-1}$ is divisible by p but not by p^2. Hence this factor must be p. Now

$$\frac{b^p-1}{b-1} = 1 + b + \ldots + b^{p-1} > b^{p-1} \geq 2^{p-1} \geq p$$

gives a contradiction.

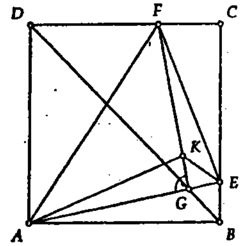

Figure 16

Answer: 135°.
Solution. Since $AGFD$ is a cyclic quadrilateral (see Fig. 16), $\angle GAF = \angle GDF = 45°$ and $\angle GFA = \angle GDA = 45°$, so triangle AGF is isosceles and $|GA| = |GF|$. Now, right triangles AGK and FGE are congruent, and $|GK| = |GE|$, so triangle GKE is also isosceles. Finally, $\angle GKE = 45°$ and $\angle EKF = 180° - \angle GKE = 135°$.

Answer: $f(x) = 1 + x$ and $f(x) = 1 - x$.
Solution. Taking $y = -f(x+1)$, we see that there is a value a such that $f(a) = 0$. We consider two cases.
Let first $a \neq 1$. Taking $y = x + 1$, we get $f(x + f(x+1)) = x + 1 + f(x+1)$. Let $g(x) = x + f(x+1)$, then $f(g(x)) = 1 + g(x)$ for all x. Since f is continuous, so is g.
Taking $y = a$ in the initial relation, we get $f(x) = a + f(x+1)$, and so $g(x-1) - g(x) = a - 1$ for all x. Since $a \neq 1$, g is unbounded and by continuity, takes all real values, so $f(z) = 1 + z$ for all z.
Let now $a = 1$, i.e., $f(1) = 0$. Then $x = 0$ yields $f(f(y)) = y$ for all reals y. Taking now $y = f(1-x)$ in the initial relation, we get $f(x + f(f(1-x))) = f(1-x) + f(x+1)$, or $0 = f(1-x) + f(x+1)$. Finally, taking $y = 1 - x$ yields $f(x + f(1-x)) = 1 - x + f(x+1)$, so $f(x + f(1-x)) = 1 - x - f(1-x)$. Let $h(x) = x + f(1-x)$, then $f(h(x)) = 1 - h(x)$ holds for all x. Replacing x with $-x$ and taking $y = 1$ in the initial relation, we get $f(-x) = 1 + f(1-x)$, so $h(x+1) - h(x) = 2$. Again, h is continuous and must take all real values, so $f(z) = 1 - z$ for all z.
It is straightforward to verify that both solutions indeed satisfy the initial relation.

Answer: 81.
Solution. By always choosing the first line, the first column and a square of the remaining 9×9 grid as the lower right square, the whole grid can be coloured in 81 moves.
We now prove that it is not possible to make more than 81 moves. Consider a sequence of moves. Select for each move one square that is chosen for the first time during this move and colour the remaining squares already before starting the sequence. Then, take all squares that were *not* selected and colour them in advance, i.e., already before starting the sequence of moves. Since all selected squares must be different, every move in the sequence now colours exactly one square.
Next, consider a bipartite graph with the 10 rows and 10 columns as vertices. Every time a square is coloured, draw an edge between the row and the column corresponding to this square. We claim that the graph is connected *before* we start the sequence of moves. Indeed, suppose that during some move, we pick rows (a, b) and columns (c, d), such that only the square (b, d) is coloured for the first time, i.e., we add the edge (b, d). But then b is already connected with d through $b - c - a - d$, so the number of connected components does not decrease. Since the graph of a fully coloured grid is connected, it must also be connected in the beginning. But a connected graph with 20 vertices must have at least 19 edges, so we can add only $100 - 19 = 81$ new edges, and hence any sequence can have at most 81 moves.