Mathematical Excalibur

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Olympiad Corner

The 19th Balkan Mathematical Olympiad was held in Antalya, Turkey on April 27, 2002. The problems are as follow.

Problem 1. Let $A_1, A_2, \ldots, A_n$ $(n \geq 4)$ be points on the plane such that no three of them are collinear. Some pairs of distinct points among $A_1, A_2, \ldots, A_n$ are connected by line segments in such a way that each point is connected to three others. Prove that there exists $k > 1$ and distinct points $X_1, X_2, \ldots, X_{2k}$ in $\{ A_1, A_2, \ldots, A_n \}$ so that for each $1 \leq i \leq 2k-1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1.

Problem 2. The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_1 = 20$, $a_2 = 30$, $a_{n+2} = 3a_{n+1} - a_n$, $n > 1$. Find all positive integers n for which $1+5a_na_{n+1}$ is a perfect square.

Problem 3. Two circles with different radii intersect at two pints A and B. The common tangents of these circles are MN and ST where the points M, S are on one of the circles and N, T are on the other. Prove that the orthocenters of the triangles AMN, AST, BMN and BST are the vertices of a rectangle.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 26, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Mathematical Games (II)

Kin Y. Li

There are many mathematical game problems involving strategies to win or to defend. These games may involve number theoretical properties or combinatorial reasoning or geometrical decomposition. Some games may go on forever, while some games come to a stop eventually. A winning strategy is a scheme that allows a player to make moves to win the game no matter how the opponent plays. A defensive strategy cuts off the opponent's routes to winning. The following examples illustrate some standard techniques.

Examples 1. There is a table with a square top. Two players take turn putting a dollar coin on the table. The player who cannot do so loses the game. Show that the first player can always win.

Solution. The first player puts a coin at the center. If the second player can make a move, the first player can put a coin at the position symmetrically opposite the position the second player placed his coin with respect to the center of the table. Since the area of the available space is decreasing, the game must end eventually. The first player will win.

Example 2. (*Bachet's Game*) Initially, there are n checkers on the table, where $n > 0$. Two persons take turn to remove at least 1 and at most k checkers each time from the table. The last person who can remove any checker wins the game. For what values of n will the first person have a winning strategy? For what values of n will the second person have a winning strategy?

Solution. By testing small cases of n, we can easily see that if n is *not* a multiple of $k + 1$ in the beginning, then the first person has a winning strategy, otherwise the second person has a winning strategy. To prove this, let n be the number of checkers on the table. If $n = (k+1)q + r$ with $0 < r < k + 1$, then the first person can win by removing r checkers each time. (Note $r > 0$ every time at the first person's turn since in the beginning it is so and the second person starts with a multiple of $k + 1$ checkers each time and can only remove 1 to k checkers.)

However, if n is a multiple of $k + 1$, then no matter how many checkers the first person takes, the second person can now win by removing r checkers every time.

Example 3. (*Game of Nim*) There are 3 piles of checkers on the table. The first, second and third piles have x, y and z checkers respectively in the beginning, where $x, y, z > 0$. Two persons take turn choosing one of the three piles and removing at least one to all checkers in that pile each time from the table. The last person who can remove any checker wins the game. Who has a winning strategy?

Solution. In base 2 representations, let

$$x = (a_1a_2\ldots a_n)_2,\quad y = (b_1b_2\ldots b_n)_2,$$
$$z = (c_1c_2\ldots c_n)_2,\quad N = (d_1d_2\ldots d_n)_2,$$

where $d_i \equiv a_i + b_i + c_i \pmod 2$. The first person has a winning strategy if and only if N is not 0, i.e. not all d_i's are 0.

To see this, suppose N is not 0. The winning strategy is to remove checkers so N becomes 0. When the d_i's are not all zeros, look at the smallest i such that $d_i = 1$, then one of a_i, b_i, c_i equals 1, say $a_i = 1$. Then remove checkers from the first pile so that $x = (e_ie_{i+1}\ldots e_n)_2$ checkers are left, where $e_j = a_j$ if $d_j = 0$, otherwise $e_j = 1 - a_j$.

(For example, if $x = (1000)_2$ and $N = (1001)_2$, then change x to $(0001)_2$.) After the move, N becomes 0. So the first person can always make a move. The second person will always have $N = 0$ at his turn and making any move will result in at least one d_i not 0, i.e. $N \neq 0$. As the number of checkers is decreasing, eventually the second person cannot make a move and will lose the game.

Example 4. Twenty girls are sitting around a table and are playing a game with n cards. Initially, one girl holds all the cards. In each turn, if at least one girl holds at least two cards, one of these girls must pass a card to each of her two neighbors. The game ends if and only if each girl is holding at most one card.

(*a*) Prove that if $n \geq 20$, then the game cannot end.
(*b*) Prove that if $n < 20$, the game must end eventually.

Solution. (*a*) If $n > 20$, then by the pigeonhole principle, at every moment there exists a girl holding at least two cards. So the game cannot end.

If $n = 20$, then label the girls $G_1, G_2, \ldots, G_{20}$ in the clockwise direction and let G_1 be the girl holding all the cards initially. Define the current value of a card to be i if it is being held by G_i. Let S be the total value of the cards. Initially, $S = 20$.

Consider before and after G_i passes a card to each of her neighbors. If $i = 1$, then S increases by $-1-1+2+20=20$. If $1 < i < 20$, then S does not change because $-i-i+(i-1)+(i+1)=0$. If $i = 20$, then S decreases by 20 because $-20-20+1+19=-20$. So before and after moves, S is always a multiple of 20. Assume the game can end. Then each girl holds a card and $S = 1+2+\cdots+20 = 210$, which is not a multiple of 20, a contradiction. So the game cannot end.

(*b*) To see the game must end if $n < 20$, let's have the two girls sign the card when it is the first time one of them passes card to the other. Whenever one girl passes a card to her neighbor, let's require the girl to use the signed card between the pair if available. So a signed card will be stuck between the pair who signed it. If $n < 20$, there will be a pair of neighbors who never signed any card, hence never exchange any card.

If the game can go on forever, record the number of times each girl passed cards. Since the game can go on forever, not every girl passed card finitely many time. So starting with a pair of girls who have no exchange and moving clockwise one girl at a time, eventually there is a pair G_i and G_{i+1} such that G_i passed cards finitely many times and G_{i+1} passed cards infinitely many times. This is clearly impossible since G_i will eventually stopped passing cards and would keep on receiving cards from G_{i+1}.

Example 5. (*1996 Irish Math Olympiad*) On a 5 × 9 rectangular chessboard, the following game is played. Initially, a number of discs are randomly placed on some of the squares, no square containing more than one disc. A turn consists of moving all of the discs subject to the following rules:

(*i*) each disc may be moved one square up, down, left or right;
(*ii*) if a disc moves up or down on one turn, it must move left or right on the next turn, and vice versa;
(*iii*) at the end of each turn, no square can contain two or more discs.

The game stops if it becomes impossible to complete another turn. Prove that if initially 33 discs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 discs on the board so that the game can continue forever.

Solution. If 32 discs are placed in the lower right 4 × 8 rectangle, they can all move up, left, down, right, repeatedly and the game can continue forever.

To show that a game with 33 discs must stop eventually, label the board as shown below:

1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1

Note that there are only eight squares labeled with 3's. A disc on 1 goes to a 3 after two moves, a disc on 2 goes to a 1 or 3 immediately, and a disc on 3 goes to a 2 immediately. Thus if k discs start on 1 and $k > 8$, the game stops because there are not enough 3's to accommodate these discs after two moves. Thus we assume $k \leq 8$, in which case there are at most sixteen discs on squares with 1's or 3's at the start, and at least seventeen discs on squares with 2's. Of these seventeen discs, at most eight can move onto squares with 3's after one move, so at least nine end up on squares with 1's. These discs will not all be able to move onto squares with 3's two moves later. So the game must eventually stop.

Example 6. (*1995 Israeli Math Olympiad*) Two players play a game on an infinite board that consists of 1 × 1 squares. Players I chooses a square and marks it with an O. Then, player II chooses another square and marks with an X. They play until one of the players marks a row or a column of five consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning.

Solution: Label the squares as shown below.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	1	2	3	3	1	2	3	3	⋯
⋯	1	2	4	4	1	2	4	4	⋯
⋯	3	3	1	2	3	3	1	2	⋯
⋯	4	4	1	2	4	4	1	2	⋯
⋯	1	2	3	3	1	2	3	3	⋯
⋯	1	2	4	4	1	2	4	4	⋯
⋯	3	3	1	2	3	3	1	2	⋯
⋯	4	4	1	2	4	4	1	2	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Note that each number occurs in a pair. The 1's and 2's are in vertical pairs and the 3's and 4's are in horizontal pairs. Whenever player I marks a square, player II will mark the other square in the pair. Since any five consecutive vertical or horizontal squares must contain a pair of these numbers, so player I cannot win.

Example 7. (*1999 USAMO*) The Y2K Game is played on a 1 × 2000 grid as follow. Two players in turn write either an *S* or an *O* in an empty square. The first player who produces three consecutive boxes that spell *SOS* wins. If all boxes are filled without producing any *SOS,* then the game is a draw. Prove that the second player has a winning strategy.

Solution. Call an empty square *bad* if playing an *S* or an *O* in that square will let the next player gets *SOS* in the next move.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***January 26, 2003***.

Problem 166. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\,[a/b] - [c/a]\,[c/b] \le c\,\min\{1/a, 1/b\}.$$

Problem 167. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the sum of their digits in base 10 representation.

Problem 168. Let AB and CD be nonintersecting chords of a circle and let K be a point on CD. Construct (with straightedge and compass) a point P on the circle such that K is the midpoint of the part of segment CD lying inside triangle ABP.

Problem 169. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 11/2 times any other group.

Problem 170. (*Proposed by Abderrahim Ouardini, Nice, France*) For any (nondegenerate) triangle with sides a, b, c, let $\sum' h(a, b, c)$ denote the sum $h(a, b, c) + h(b, c, a) + h(c, a, b)$. Let $f(a, b, c) = \sum' (a / (b + c - a))^2$ and $g(a, b, c) = \sum' j(a, b, c)$, where $j(a,b,c) = (b + c - a) / \sqrt{(c + a - b)(a + b - c)}$. Show that $f(a, b, c) \ge \max\{3, g(a, b, c)\}$ and determine when equality occurs. (Here $\max\{x,y\}$ denotes the maximum of x and y.)

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Solutions

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Problem 161. Around a circle are written all of the positive integers from 1 to N, $N \ge 2$, in such a way that any two adjacent integers have at least one common digit in their base 10 representations. Find the smallest N for which this is possible. (*Source: 1999 Russian Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yan Sau** (True Light Girls' College, Form 6), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **LAM Wai Pui** (STFA Leung Kau Kui College, Form 5), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 5).

Note one of the numbers adjacent to 1 is at least 11. So $N \ge 11$. Then one of the numbers adjacent to 9 is at least 29. So $N \ge 29$. Finally $N = 29$ is possible by writing 1, 11, 12, 2, 22, 23, 3, 13, 14, 4, 24, 25, 5, 15, 16, 6, 26, 27, 7, 17, 18, 8, 28, 29, 9, 19, 21, 20, 10 around a circle. Therefore, the smallest N is 29.

Problem 162. A set of positive integers is chosen so that among any 1999 consecutive positive integers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other. (*Source: 1999 Russian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Define $A(1, i) = i$ for i=1,2,…, 1999. For $k \ge 2$, let $B(k)$ be the product of $A(k–1, 1)$, $A(k–1, 2)$, …, $A(k–1, 1999)$ and define $A(k, i) = B(k) + A(k–1, i)$ for i = 1,2,…, 1999. Since $B(k)$ is a multiple of $A(k–1, i)$, so $A(k, i)$ is also a multiple of $A(k–1, i)$. Then $m < n$ implies $A(n, i)$ is a multiple of $A(m, i)$.

Also, by simple induction on k, we can check that $A(k, 1)$, $A(k, 2)$, …, $A(k, 1999)$ are consecutive integers. So for k = 1,2, …, 2000, among $A(k, 1)$, $A(k, 2)$, …, $A(k, 1999)$, there is a chosen number $A(k, i_k)$. Since $1 \le i_k \le 1999$, by the pigeonhole principle, two of the i_k's are equal. Therefore, among the chosen numbers, there are two numbers with one dividing the other.

Comments: The condition "among any 1999 consecutive positive integers, there is a chosen number" is meant to be interpreted as "among any 1999 consecutive positive integers, there exists at least one chosen number." The solution above covered this interpretation.

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 7), **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 163. Let a and n be integers. Let p be a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be the product of two nonconstant polynomials with integer coefficients. (*Source: 1999 Romanian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School).

Assume we have $f(x) = g(x)\,h(x)$, where $g(x)$ and $h(x)$ are two nonconstant polynomials with integer coefficients. Since $p = f(0) = g(0)\,h(0)$, we have either

$$g(0) = \pm p,\quad h(0) = \pm 1$$
$$\text{or}\quad g(0) = \pm 1,\quad h(0) = \pm p.$$

Without loss of generality, assume $g(0) = \pm 1$. Then $g(x) = \pm x^m + \cdots \pm 1$. Let z_1, z_2, …, z_m be the (possibly complex) roots of $g(x)$. Since $1 = |g(0)| = |z_1|\,|z_2| \cdots |z_m|$, so $|z_i| \le 1$ for some i. Now $0 = f(z_i) = z_i^n + az_i + p$ implies

$$p = -z_i^n - az_i \le |z_i|^n + |a|\,|z_i| \le 1 + |a|,$$

a contradiction.

Other commended solvers: **FOK Kai Tung** (Yan Chai Hospital No. 2 Secondary School, Form 6).

Problem 164. Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of AC and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$. (*Source: 1999 Belarussian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Let AO cut BC at D and AP extended cut OC at E. By Ceva's theorem (ΔAOC and point P), we have

$$\frac{AM}{MC} \times \frac{CE}{EO} \times \frac{OD}{DA} = 1.$$

Since $AM = MC$, we get $OD/DA = OE/EC$, which implies $DE \| AC$. Then $\angle EDC = \angle DCA = \angle DAC = \angle ODE$, which implies DE bisects $\angle ODC$. In ΔACD, since CE and DE are external angle bisectors at $\angle C$ and $\angle D$ respectively, so E is the excenter of ΔACD opposite A. Then AE bisects $\angle OAC$ so that $\angle DAP = \angle CAP$. Finally,

$$\begin{aligned}\angle BAP &= \angle BAD + \angle DAP \\ &= \angle DCA + \angle CAP \\ &= \angle BPA.\end{aligned}$$

Therefore, $AB = BP$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13).

Problem 165. For a positive integer n, let $S(n)$ denote the sum of its digits. Prove that there exist distinct positive integers $n_1, n_2, \ldots, n_{50}$ such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{50} + S(n_{50}).$$

(*Source: 1999 Polish Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

We will prove the statement that for $m > 1$, there are positive integers $n_1 < n_2 < \cdots < n_m$ such that all $n_i + S(n_i)$ are equal and n_m is of the form $10\cdots08$ by induction.

For the case $m = 2$, take $n_1 = 99$ and $n_2 = 108$, then $n_i + S(n_i) = 117$.

Assume the case $m = k$ is true and $n_k = 10\cdots08$ with h zeros. Consider the case $m = k + 1$. For $i = 1, 2, \ldots, k$, define

$$N_i = n_i + C, \text{ where } C = 99\cdots900\cdots0$$

(C has $n_k - 8$ nines and $h + 2$ zeros) and $N_{k+1} = 10\cdots08$ with $n_k - 7 + h$ zeros. Then for $i = 1, 2, \ldots, k$,

$$N_i + S(N_i) = C + n_i + S(n_i) + 9(n_k - 8)$$

are all equal by the case $m = k$. Finally,

$$\begin{aligned}N_k + S(N_k) &= C + n_k + 9 + 9(n_k - 8) \\ &= 10\cdots017 \ (n_k - 8 + h \text{ zeros}) \\ &= 10\cdots008 + 9 \\ &= N_{k+1} + S(N_{k+1})\end{aligned}$$

completing the induction.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5).

Olympiad Corner

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Problem 4. Find all functions $f: \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$,

$$\begin{aligned}2n + 2001 \le\ & f(f(n)) + f(n) \\ \le\ & 2n + 2003.\end{aligned}$$

($\mathbb{N}$ is the set of all positive integers.)

Mathematical Games II

(continued from page 2)

<u>Key Observation</u>: A square is bad if and only if it is in a block of 4 consecutive squares of the form $S{*}{*}S$, where * denotes an empty square.

(<u>*Proof.*</u> Clearly, the empty squares in $S{*}{*}S$ are bad. Conversely, if a square is bad, then playing an O there will allow an SOS in the next move by the other player. Thus the bad square must have an S on one side and an empty square on the other side. Playing an S there will also lose the game in the next move, which means there must be another S on the other side of the empty square.)

Now the second player's winning strategy is as follow: after the first player made a move, play an S at least 4 squares away from either end of the grid and from the first player's first move. On the second move, the second player will play an S three squares away from the second player's first move so that the squares in between are empty. (If the second move of the first player is next to or one square away from the first move of the second player, then the second player will place the second S on the other side.) After the second move of the second player, there are 2 bad squares on the board. So eventually somebody will fill these squares and the game will not be a draw.

On any subsequent move, when the second player plays, there will be an odd number of empty squares and an even number of bad squares, so the second player can always play a square that is not bad.

Example 8. (*1993 IMO*) On an infinite chessboard, a game is played as follow. At the start, n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece that has been jumped over is then removed. Find those values of n for which the game can end with only one piece remaining on the board.

Solution. Let $\mathbb{Z}$ denotes the set of integers. Consider the pieces placed at the *lattice points* $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$. For $k = 0, 1, 2$, let $C_k = \{(x, y) \in \mathbb{Z}^2 : x+y \equiv k \pmod 3\}$. Let a_k be the number of pieces placed at lattice points in C_k.

A horizontal move takes a piece at (x, y) to an unoccupied point $(x \pm 2, y)$ jumping over a piece at $(x \pm 1, y)$. After the move, each a_k goes up or down by 1. So each a_k changes parity. If n is divisible by 3, then $a_0 = a_1 = a_2 = n^2/3$ in the beginning. Hence at all time, the a_k's are of the same parity. So the game cannot end with one piece left causing two a_k's 0 and the remaining 1.

If n is not divisible by 3, then the game can end. We show this by induction. For $n = 1$ or 2, this is easily seen. For $n \ge 4$, we introduce a trick to reduce the $n \times n$ square pieces to $(n-3) \times (n-3)$ square pieces.

<u>*Trick:*</u> Consider pieces at (0,0), (0,1), (0,2), (1,0). The moves $(1,0) \to (-1,0)$, $(0,2) \to (0,0)$, $(-1,0) \to (1,0)$ remove three consecutive pieces in a column and leave the fourth piece at its original lattice point.

We can apply this trick repeatedly to the $3 \times (n-3)$ pieces on the bottom left part of the $n \times n$ squares from left to right, then the $n \times 3$ pieces on the right side from bottom to top. This will leave $(n-3) \times (n-3)$ pieces. Therefore, the $n \times n$ case follows from the $(n-3) \times (n-3)$ case, completing the induction.

19th Balkan Mathematical Olympiad

Antalya, Turkey, April 27, 2002

Problem 1

Let $A_1, A_2, ..., A_n$ $(n \geq 4)$ be points on the plane such that no three of them are collinear. Some pairs of distinct points among $A_1, A_2, ..., A_n$ are connected by line segments in such a way that each point is connected to at least three others. Prove that there exists $k > 1$ and distinct points $X_1, X_2, ..., X_{2K}$ in $\{A_1, A_2, ..., A_n\}$ so that for each $1 \leq i \leq 2k-1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1.

Solution

If two points are connected by a segment, we say they are **adjacent**. Let $P = X_1X_2...X_s$, with $X_i \in \{A_1, ..., A_n\}$, $X_i \neq X_j$, $i \neq j$, be a longest sequence such that X_i and X_{i+1} adjacent for each $i = 1, ..., s-1$. According to the assumption X_1 is adjacent to at least three points. Two of them, say Y and Z, most be distinct from X_2. On the other hand $\{Y, Z\} \subset \{X_3, ..., X_s\}$ by maximality of P. Let $Y = X_i$ and $Z = X_j$, $i < j$. Then $C = X_1X_2 ... X_i .. X_jX_1$ is a cycle with a chord X_1X_i.

If j is even, then C is desired even cycle. Otherwise J is odd and one of the cycles (exactly one) $X_1X_2 ... X_iX_j$ and $X_1X_iX_{i+1}..X_jX_1$ is even. This completes the solution.

Problem 2

The sequence $a_1, a_2, ..., a_n, ...$ is defined by

$$a_1 = 20, \quad a_2 = 30, \quad a_{n+2} = 3a_{n+1} - a_n, \qquad n > 1.$$

Find all positive integers n for which $1 + 5a_na_{n+1}$ is a perfect square.

Solution

If $b_n = a_{n+1} + b_n$ and $c_n = 1 + 5a_na_{n+1}$, then

$$5a_{n+1} = b_{n+1} + b_n, \quad a_{n+2} - a_n = b_{n+1} - b_n$$

and hence

$$c_{n+1} - c_n = 5a_{n+1}(a_{n+2} - a_n) = b_{n+1}^2 - b_n^2.$$

Therefore, for any n, we have,

$$c_{n+1} - b_{n+1}^2 = c_n - b_n^2 = c_1 - b_1^2 = 5001 = 3 \cdot 167.$$

Let $c_n = m^2$ for some positive integers n and m. Since 3 and 167 are prime numbers, it follows that $m + b_n = 167$, $m + b_n = 3$, or $m + b_n = 501$, $m - b_n = 1$. Hence $c_n = 85^2$, or $c_n = 251^2$. On the other hand, a simple induction shows that the sequence $\{c_n\}_{n=1}^{\infty}$, is strictly monotone.

Since $c_1 = 1 + 5 \cdot 20 \cdot 30 < 84^2 < c_2 = 1 + 5 \cdot 30 \cdot 70 < c_3 = 1 + 5 \cdot 70 \cdot 180 = 251^2$, the only solution of the problem is $n = 3$,.

Problem 3

Two circles with different radii intersect at two points A and B. The common tangents of these circles are MN and ST whwre the points M, S are on one of the circles and N, T are on the other.

Prove that the orthocenters of the triangles AMN, AST, BMN and BST are the vertices of a rectangle.

Solution

By a reflection with respect to the line of the centers (d) the for points H_1, H_2, H_3, H_4 the orthocenters of the given triangles, form an isosceles trapezium.

It will be sufficient to prove that H_1B is perpendicular on AB.

Let Q, P be the intersection points of MN with AB respectively AH_1. Q is then the midpoint of MN and we are left to prove that H_1BQP is cyclic, or that is the same to prove $AH_1 \cdot AP = AB \cdot AQ$. (1)

The power of the point P with respect to the circle (AMN) gives:

$$PM \cdot PN = PH_1 \cdot PA. \qquad (2)$$

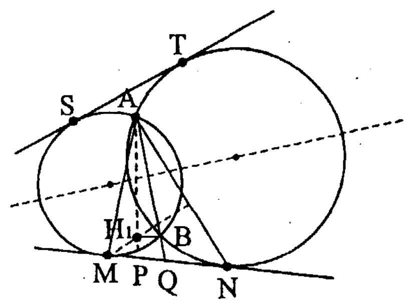

Fig. 8

Hence

$$\begin{aligned} AH_1 \cdot AP &= (AP - H_1P)AP = AP^2 - PM \cdot PN \\ &= (AQ^2 - PQ^2) - (MQ - PQ)(NQ + PQ) \\ &= AQ^2 - PQ^2 - MQ^2 + PQ^2 = AQ^2 - AQ \cdot BQ = AQ(AQ - BQ) \\ &= AQ \cdot AB, \end{aligned}$$

which proves the desired result

Problem 4

Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$

$$2n + 2001 \le f(f(n)) + f(n) \le 2n + 2003.$$

($\mathbb{N}$ is the set of all positive integers).

Solution

We define a sequence of non-negative integers as follows: $a_0 \in \mathbb{N}$, $a_{k+1} = f(a_k)$, for every $k \in \mathbb{N}$. Replacing in the above relation n with a_k, we obtain: $2001 \le a_k + a_{k+1} - 2a_{k-2} \le 2003$, for all $n \ge 2$.

Let $b_k = a_k - 667k$. It follows: $b_k + b_{k-1} - 2b_{k-2} = a_k + a_{k-1} - 2a_{k-2} - 2001$.

The inequality $0 < b_k + b_{k-1} - 2b_{k-2} \le 2$ can be written as:

$$b_k - b_{k-1} = -2(b_{k-2} - b_{k-2}) + \varepsilon_k,$$

where ε_k can take only the values 0, 1 or 2, so we obtain:

$$\begin{aligned} b_k - b_{k-1} &= -2(b_{k-1} - b_{k-2}) + \varepsilon_k = (-2)^2(b_{k-2} - b_{k-3}) + (-2)\varepsilon_{k+1} + \varepsilon_k \\ &= \ldots = (-2)^{k-1}(b_1 - b_0) + (-2)^{k-2}\varepsilon_2 + (-2)^{k-3} + \ldots + \varepsilon_k. \end{aligned}$$

Adding up the these equalities we obtain:

$$b_n = \frac{2b_0 + b_1}{3} + (-2)^n \frac{b_1 - b_0}{3}$$

$$+ \frac{2}{3}(\varepsilon_n + (-2)\varepsilon_{n-1} + (-2)^2\varepsilon_{n-2} + \ldots + (-2)^{n-2}\varepsilon_2) + \frac{1}{3}(\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_n).$$

As $\varepsilon_k \in \{0,1,2\}$ the maximum of the sum that contains the powers of -2 can be realized when positive terms are considered and with $\varepsilon_k = 2$. Thus:

$$\varepsilon_n + (-2)\varepsilon_{n-1} + (-2)^2\varepsilon_{n-2} + \ldots + (-2)^{n-2}\varepsilon_2 \le 2(1 + 2^2 + 4^4 + \ldots + 2^{2\left[\frac{n-2}{2}\right]}) =$$

$$\frac{2}{3}(2^{2\left[\frac{n-2}{2}\right]+2} - 1).$$

We conclude that there are α, β, γ with $\gamma = b_1 - b_0$, such that, for any n:

$$b_n \le \alpha + \beta_n + \frac{(-2)^n}{3}\gamma + \frac{2^{2\left[\frac{n-2}{2}\right]+3}}{9}.$$

As $b_n = \alpha_n - 667n$, with the notation $\beta' = \beta + 667$ we have:

$$\alpha_n \le \alpha + \beta'_n + \frac{(-2)^n}{3}\gamma + \frac{2^{2\left[\frac{n-2}{2}\right]+3}}{9}.$$

We claim that $\gamma=0$. Indeed, suppose $\gamma>0$. For odd values of n:

$$\alpha_n \le \alpha + \beta'_n + \frac{2^n}{3}(\gamma + \frac{2}{3}).$$

As γ is an integer have $\gamma \ge 0$ so that as $2^n > n^2$ for $n > 4$ we have

$$\alpha_n \le \alpha + \beta'_n - \frac{1}{9}n^2,$$

and the quadratic function $q(n) = \alpha + \beta'_n - \frac{2}{3}n^2$ eventually takes strictly negative values. The case $\gamma<0$ is similar if we consider even values for n.

In conclusion $b_1 = b_0$, hence $\alpha_1 = \alpha_0 + 667$, that is $f(\alpha_0) = \alpha_0 + 667$. Since α_0 was arbitrarily chosen, it follows that $f(x) = x + 667$ for every $x \in \mathbb{N}$.

Finally, observe that the preceding functions fulfills the conditions of the problem.