Mathematical Excalibur

Volume 3, Number 4 September-November, 1997

Olympiad Corner

British Mathematical Olympiad:

Round 1 (January 15, 1997)
Time Allowed: $3\frac{1}{2}$ hours.

Problem 1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$. Determine all such integers N for which $R(N) = 4N + 3$.

Problem 2. For positive integers n, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by

$$a_1 = 1,\ a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \cdots + a_{n-1}),\ n > 1.$$

Determine the value of a_{1997}.

Problem 3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.
In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

(*continued on page 4*)

Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is January 10, 1998.

For individual subscription for the four remaining issues for the 97-98 academic year, send us four stamped self-addressed envelopes. Send all correspondence to:

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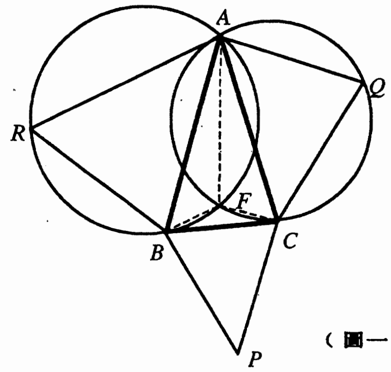
老師不教的幾何（四）

張百康

在任意的三角形的三邊上作另一些三角形，只要滿足一些簡單的條件，卻常常可以得到一些美妙的結果。

圖一的三角形 ABC 的三條邊外側分別隨意作了三個三角形 ABR、BCP 和 CAQ，並同時作三角形 ABR 和 CAQ 的外接圓。連此兩外接圓的一交點 F 至 A、B 及 C。



（圖一）

$$\begin{aligned}\angle BFC &= 360° - \angle AFB - \angle AFC \\ &= 360° - (180° - \angle ARB) \\ &\qquad - (180° - \angle AQC) \\ &= \angle ARB + \angle AQC.\end{aligned}$$

如果條件

$$\angle BPC + \angle ARB + \angle CQA = 180°$$

成立，$\angle BPC$ 和 $\angle BFC$ 互補，因此三角形 BCP 的外接圓也通過點 F。這個條件並不難得，下列兩種情況都是它的特例：

(1) 三角形 ABR、CPB 和 QCA 相似；

(2) A、B 和 C 分別是三角形 PQR 的邊 QR、RP 和 PQ 上的點。

如果三角形 ABR、CPB 和 QCA 相似，則它們的外接圓心 O_3、O_1 和 O_2 所組成的三角形也和它們相似（圖二），道理如下：

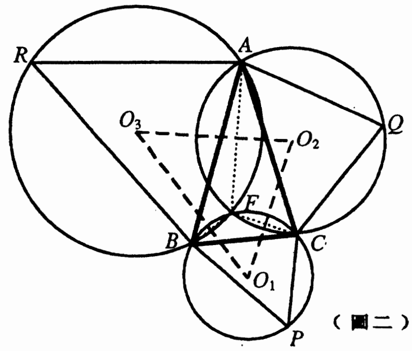
由於 O_1O_2 和 O_1O_3 是圓心連線，所以它們分別垂直公共弦 CF 和 BF，因此

$$\begin{aligned}\angle O_2O_1O_3 &= 360° - 90° - 90° - \angle BFC \\ &= 180° - \angle BFC \\ &= \angle CPB\ (=\angle ABR=\angle QCA)。\end{aligned}$$

同理

$$\angle O_1O_3O_2 = \angle BRA\ (=\angle CAQ=\angle PBC),$$
$$\angle O_3O_2O_1 = \angle AQC\ (=\angle RAB=\angle BCP),$$

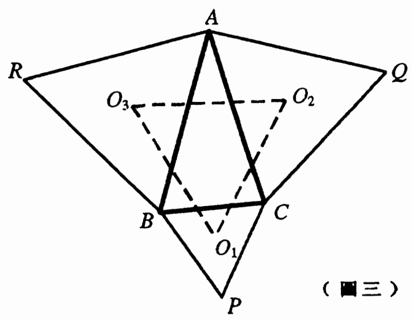
證畢。



（圖二）

拿破侖(Napoleon)是一位大家都知道的大將軍，但你可知道他對數學，尤其是幾何，有濃厚興趣？我現在要介紹的一類三角形，據說是他發現的，所以後人將這種三角形命名為拿破侖三角形。

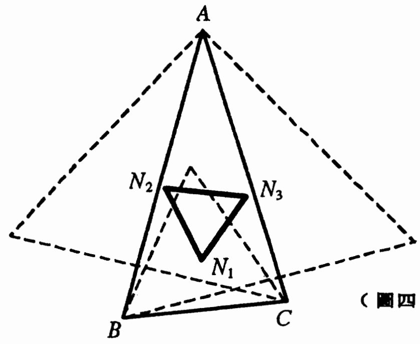
在任意的一個三角形 ABC 的三條邊上，分別向外側作三等邊三角形 ABR、BCP 和 CAQ（圖三）。這三個等邊三角形的心 O_1、O_2、O_3 可連成一三角形 $O_1O_2O_3$，稱



（圖三）

為外拿破侖三角形。由前述結果可推知外拿破侖三角形也是等邊三角形。

如果我們改變一下上述作法，把三個等邊三角形作於三角形 ABC 三條邊的內側，可以得到如圖四所示的另一三角形 $N_1N_2N_3$，稱為內拿破侖三角形。



（圖四）

俄羅斯數學家I.M. Yaglom 巧妙地證明內拿破侖三角形也是等邊三角形：

應用餘弦公式於圖三的三角形 AO_3O_2 可得

$$(O_2O_3)^2 = \frac{b^2}{3} + \frac{c^2}{3} - 2\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}\cos(A+60°),$$

這裏我們利用了

$$AO_2 = \frac{b}{\sqrt{3}}\text{，}AO_3 = \frac{c}{\sqrt{3}}$$

和 $\angle O_3AO_2 = A + 60°$

等簡單事實，請同學們自行驗證。

類似手法再應用於三角形 AN_3N_2 可得

$$(N_2N_3)^2 = \frac{b^2}{3} + \frac{c^2}{3} - 2\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}\cos(60°-A).$$

將上述兩等式同側相減可得

$$\begin{aligned}(O_2O_3)^2 - (N_2N_3)^2 &= \frac{2bc}{3}\left(\cos(60°-A) - \cos(A+60°)\right)\\ &= \frac{2}{\sqrt{3}}bc\sin A\\ &= \frac{4}{\sqrt{3}}\times \Delta ABC \text{ 的面積。}\end{aligned}$$

此處的簡化過程從略。

由於 $O_2O_3 = O_3O_1 = O_1O_2$，因此 $N_2N_3 = N_3N_1 = N_1N_2$。這證明的巧

(*continued on page 4*)

Inverse Sequences and Complementary Sequences

Yau Kwan Kiu Garry
Form 7, Queen's College

Editor's Note: This article is modified and shortened by the editors.

Consider the sequence

$$f(n) = 0, 0, 0, 1, 2, 3, 3, 4, 5, 6, 7, 10, \ldots,$$

i.e., $f(1) = 0, f(2) = 0, f(3) = 0, f(4) = 1$, etc. We can construct another sequence $f^*(n)$ according to the definition

$$f^*(n) = k, \text{ where } f(k) < n \le f(k+1).$$

For our example,

$$f^*(n) = 3, 4, 5, 7, 8, 9, 10, 11, 11, 11, \ldots.$$

Note that $f^*(n)$ can also be referred as the "frequency distribution function" of $f(n)$ since $f^*(n)$ is the number of terms in the sequence f that are less than n.

Figure 1 shows the two functions $f(n)$ and $f^*(n)$. We note something interesting: f^* is a mirror image of f. If we compute the frequency distribution of $f^*(n)$, we obtain $f(n)$ again. That is, $f^{**}(n) = f(n)$. The sequences $f(n)$ and $f^*(n)$ are called inverse sequences.

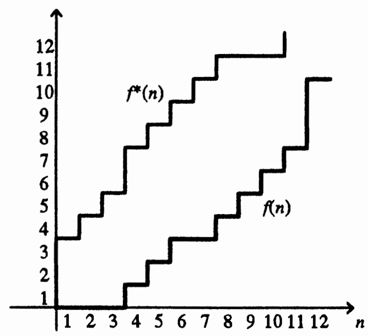


Figure 1. The functions $f(n)$ and $f^*(n)$.

Now we construct two other sequences

$$F(n) = f(n) + n \text{ and } G(n) = f^*(n) + n.$$

For our example,

$$F(n) = 1, 2, 3, 5, 7, 9, 10, 12, 14, \ldots;$$
$$G(n) = 4, 6, 8, 11, 13, 15, 17, 19, 20, \ldots.$$

Notice anything? The two sequences $F(n)$ and $G(n)$ together contain each natural number exactly once. This fact and its converse were first discovered and proved by mathematicians Lambek and Moser in 1954 (c.f. American Mathematical Monthly, vol. 61, p. 454, 1954). The sequences $F(n)$ and $G(n)$ are called complementary sequences.

Theorem (Lambek and Moser). $f(n)$ and $f^*(n)$ are inverse sequences if and only if $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$ are complementary sequences (with the minor conditions that (i) $f(n)$ and $f^*(n)$ are non-decreasing sequences of non-negative integers; (ii) $F(n)$ and $G(n)$ are strictly increasing sequences of positive integers.)

If a formula for the nth term of a sequence is known, the theorem of Lambek and Moser can be used to find a general formula for the complementary sequence. The following example illustrates the idea.

Example. We can separate the natural numbers into two sequences $F(n)$ and $G(n)$ that contain squares and non-squares as follows.

$$F(n) = 1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots,$$
$$G(n) = 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, \ldots.$$

We know that a formula for the nth square is $F(n) = n^2$. Can we find a formula for the nth non-square $G(n)$?

We note that $F(n)$ and $G(n)$ are complementary and thus the sequences

$$f(n) = F(n) - n = 0, 2, 6, 12, 20, \ldots,$$
$$f^*(n) = G(n) - n = 1, 1, 2, 2, 2, 2, 3, \ldots,$$

are inverse sequences. Now

$$f(n) = F(n) - n = n^2 - n.$$

Therefore, $f^*(n) = k$ where

$$f(k) < n \le f(k+1),$$
$$k^2 - k < n \le (k+1)^2 - (k+1) = k^2 + k.$$

Since both k and n are integers,

$$k^2 - k + \tfrac{1}{4} < n < k^2 + k + \tfrac{1}{4},$$
$$(k - \tfrac{1}{2})^2 < n < (k + \tfrac{1}{2})^2,$$
$$k - \tfrac{1}{2} < \sqrt{n} < k + \tfrac{1}{2},$$
$$\sqrt{n} - \frac{1}{2} < k < \sqrt{n} + \frac{1}{2}.$$

Consequently,

$$f^*(n) = k = \left[\sqrt{n} + \tfrac{1}{2}\right]$$

and

$$G(n) = f^*(n) + n = n + \left[\sqrt{n} + \tfrac{1}{2}\right].$$

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is January 10, 1998.

Problem 66.

(a) Find the first positive integer whose square ends in three 4′s.

(b) Find all positive integers whose squares end in three 4′s.

(c) Show that no perfect square ends with four 4′s.

(*Source:* 1995 British Mathematical Olympiad.)

Problem 67. Let Z and R denote the integers and real numbers, respectively. Find all functions $f: Z \to R$ such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{2}$$

for all integers x, y such that $x + y$ is divisible by 3. (*Source:* a modified problem from the 1995 Iranian Mathematical Olympiad.)

Problem 68. If the equation

$$ax^2 + (c-b)x + (e-d) = 0$$

has real roots greater than 1, show that the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has at least one real root. (*Source*: 1995 Greek Mathematical Olympiad.)

Problem 69. $ABCD$ is a quadrilateral such that $AB = AD$ and $\angle B = \angle D = 90°$. Points F and E are chosen on BC and CD, respectively, so that $DF \perp AE$. Prove that $AF \perp BE$. (*Source:* 1995 Russian Mathematical Olympiad.)

Problem 70. Lines $l_1, l_2, \cdots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \cdots, k-1$ so that in each of the lines $l_1, l_2, \cdots, l_k$ the numbers $1, 2, \cdots, k-1$ appear exactly once if and only if k is even. (*Source:* a modified problem from the 1995 Greek Mathematical Olympiad.)

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Solutions

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Due to the large number of solutions received by the editors, we will first acknowledge the solvers by their schools and grade levels. The numbers following a solver's name are the number of the problems which the solver submitted correct solutions.

Bishop Hall Jubilee School: (Form 4) CHAN Kin Hang (61, 63, 64, 65). **Cheung Chuk Shan College**: (Form 5) CHOW King Fun (61). **Heep Woh College**: (Form 7) KU Wah Kwan (61, 63). **Ho Fung College**: (Form 6) TSE Wing Ho (61, 64). **HK Taoist Association Ching Chung Secondary School**: (Form 7) LI Fung (61, 62). **HKUST**: CHAN Wing Sum (61, 63). **La Salle College**: (Form 3) CHAN Ernest Eason (61); (Form 5) Vincent LUNG (61). **N.T. Heung Yee Kuk Yuen Long District Secondary School**: (Form 7) CHU Kai Mun (61, 63, 64). **Queen Elizabeth School**: (Form 4) LAI Chi Fung Brian (61), LAW Ka Ho (61, 62, 63, 64, 65). **Saint Louis School**: (Form 7) SHAM Wing Hang (61). **St. Paul's Co-educational College**: (Form 5) CHAN Lung Chak (61, 62), MAK Shiu Ting (61), NGAN Chung Wai Hubert (61, 62, 63, 64, 65), SHEK Ka Wai Wilson (62); (Form 7) CHU Choi Yam Venus (61). **St. Stephen's Girls' College**: (Form 6) WAN Hoi Wah (61). **SKH Kei Hau Secondary School**: (Form 4) WONG Chun Wai (61, 62, 63, 64, 65). **Shi Hui Wen Secondary School**: (Form 6) Jimmy KONG Ka Ho (61, 62, 64, 65). **STFA Leung Kau Kui College**: (Form 5) CHU Chun Yiu (61, 63), IP Man Wai (61), Gary NG Ka Wing (61, 62, 63, 64, 65), SIN Ka Fai (61, 62, 64), YUEN Man Long (61, 62, 63, 64, 65); (Form 6) Yves CHEUNG Yui Ho (61, 62, 63, 64), CHING Wai Hung (61, 62, 64), WONG Hau Lun (61, 62, 63, 64, 65); (Form 7) William CHEUNG Pok Man (62, 63, 64). **Valtorta College**: (Form 6) CHANG Pui Kwan (61), KO Tsz Wan (61), Ryan LAI (61), LAM Wai Hung (61), LIN Kai Shuen (61), NG Lai Ha (61), TAM Ka Kwong (61), TANG Ka Wai (61), WONG Shu Fai (61); (Form 7) KWAN Yee Kin (61), LEUNG Pak Keung (62), TSANG Sai Wing (62), WAN Tsz Kit (61, 62, 64).

Problem 61. Find the smallest positive integer which can be written as the sum of nine, the sum of ten and the sum of eleven consecutive positive integers.

Solution:

Let n be the smallest such positive integer. Then

$$n = a + (a+1) + \cdots + (a+8) = 9a + 36,$$
$$n = b + (b+1) + \cdots + (b+9) = 10b + 45,$$
$$n = c + (c+1) + \cdots + (c+10) = 11c + 55.$$

These imply n is divisible by

$$9 \times 5 \times 11 = 495.$$

So $n \geq 495$. Letting $a = 51$, $b = 45$, $c = 40$, we see that 495 is possible. So $n = 495$.

Problem 62. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the midpoint of PQ. (*Source*: 1996 Australian Mathematical Olympiad.)

Solution: WONG Chun Wai.

Let $[XYZ]$ denote the area of ΔXYZ. Then

$$\frac{MP}{MQ} = \frac{[PAC]}{[QAC]} = \frac{\frac{AP}{AB}[ABC]}{\frac{AQ}{AD}[ADC]} = \frac{CD \cdot AD \cdot [ABC]}{AB \cdot BC \cdot [ADC]} = \frac{[ADC] \cdot [ABC]}{[ABC] \cdot [ADC]} = 1.$$

Problem 63. Show that for $n \geq 2$, there is a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $|a_k - k| = |a_1 - 1| \neq 0$ for $k = 2, 3, \ldots, n$ if and only if n is even.

Solution: LAW Ka Ho.

Suppose for some n, the condition is possible. Let $d = |a_1 - 1|$, p be the number of times $a_k > k$ and q be the number of times $a_k < k$. Then $p + q = n$ and

$$0 = (a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = pd - qd.$$

So $p = q$ and n is even. If n is even, then the permutation $2, 1, 4, 3, \ldots, n, n-1$ satisfies the condition with $|a_1 - 1| = 1$.

Comments: This was a problem on the 1996 Australian Mathematical Olympiad.

Problem 64. Show that it is impossible to place 1995 different positive integers

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

along a circle so that for every two adjacent numbers, the ratio of the larger to the smaller one is a prime number.

Solution: William CHEUNG Pok Man.

Suppose this is possible. Let a_1, a_2, …, a_{1995} be the numbers in the clockwise direction. Then a_{k-1}/a_k is a prime or the reciprocal of a prime for k = 1, 2, …, 1995 with $a_0 = a_{1995}$. Suppose m of these are primes and 1995 − m of these are reciprocals of primes. Since

$$\left(\frac{a_0}{a_1}\right)\left(\frac{a_1}{a_2}\right)\cdots\left(\frac{a_{1994}}{a_{1995}}\right)=1,$$

this means the product of m primes will equal to a product of 1995 − m primes. Unique prime factorization implies m = 1995 − m, which is impossible as 1995 is odd.

Comments: This was a problem on the 1995 Russia Mathematical Olympiad.

Problem 65. All sides and diagonals of a regular 12-gon are painted in 12 colors (each segment is painted in one color). Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

Solution: LAW Ka Ho.

There are 12 sides and 54 diagonals. With 12 colors, there is a color, say X, which is used to paint at most 5 of these segments. For each X colored segment, 10 triangles can be formed having this segment as a side (using the remaining 10 vertices). So there are at most 50 triangles with at least one side colored X. However, if any three colors are the colors of the sides of a triangle, there would be C_2^{11} = 55 triangles having at least one side colored X, a contradiction.

Comments: This was also a problem on the 1995 Russia Mathematical Olympiad.

Olympiad Corner

(*continued from page 1*)

Problem 4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

Problem 5. Let x, y and z be positive real numbers.

(i) If $x + y + z \geq 3$, is it necessarily true that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq 3$?

(ii) If $x + y + z \leq 3$, is it necessarily true that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3$?

Round 2 (February 27, 1997)
Time Allowed: $3\frac{1}{2}$ hours.

Problem 1. Let M and N be two 9-digit positive integers with the property that if any one digit of M is replaced by the digit of N in the corresponding place (e.g., the 'tens' digit of M replaced by the 'tens' digit of N) then the resulting integer is a multiple of 7.

Prove that any number obtained by replacing a digit of N by the corresponding digit of M is also a multiple of 7.

Find an integer $d > 9$ such that the above result concerning divisibility by 7 remains true when M and N are two d-digit positive integers.

Problem 2. In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

Problem 3. Find the number of polynomials of degree 5 with distinct coefficients from the set {1, 2, 3, 4, 5, 6, 7, 8} that are divisible by $x^2 - x + 1$.

Problem 4. The set

$$S = \{1/r : r = 1, 2, 3, \ldots\}$$

of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, 1/20, 1/8, 1/5 is such a progression, of length 3 (and common difference 3/40). Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S (−1/40 and 11/40 not being members of S).

(i) find a maximal progression in S of length 1996.

(ii) Is there a maximal progression in S of length 1997?

老師不教的幾何（四）

(*continued from page 2*)

妙處在於它帶給我們另一個美麗而意想不到的結果：

外拿破侖三角形的面積
− 內拿破侖三角形的面積
= 三角形ABC的面積，

同學們請自己驗證便可。

在任意三角形ABC的三邊外側作等邊三角形後，還有另一個美妙的特性是十七世紀數學家費馬(Fermat) 所發現的：

圖五中三角形ABR 、BCP 和CAQ都是等邊的，所以ΔARC 繞點A旋轉 60°可得ΔABQ。因此

$$RC = BQ$$

及 $$\angle RFB = 60°。$$

同理，$PA = CR$。所以

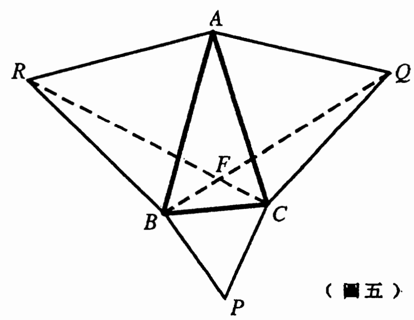
$$AP = BQ = CR。$$

再者，

$$\angle RFB = 60° = \angle RAB$$

及

$$\angle CFQ = 60° = \angle CAQ。$$



（圖五）

因此$ARBF$ 和$CQAF$ 都是圓外接四邊形。由於$\angle BFC$ = 120°，而$\angle CPB$ = 60°，可以推知$BPCF$ 也是圓外接四邊形。這三個圓於F 共點，稱為費馬點。由F 原是BQ和CR 的交點，從對稱觀點可知F 也在AP上。